Norwegian University of Science and Technology
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English

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EXAM IN COURSE TMA4245 STATISTICS

May 15 2009
Tid: 09:00–13:00

Permitted aids: *Tabeller og formler i statistikk*, Tapir Forlag
K. Rottmann: *Matematisk formelsamling*
Calculator HP30S / CITIZEN SR-270X
Yellow, stamped A5-sheet with your own handwritten notes.

Examination results are due: June 9 2009

Problem 1 The waterworks

A waterworks has two pumps for pumping water from a water drinking source to a water-reservoir. In order to be able to pump water one of the pumps has to function. On an arbitray day let A_1 be the event that pump 1 is functioning and let A_2 be the event that pump 2 is functioning. From experience we know that $P(A_1) = 0.8$, $P(A_2) = 0.8$ and $P(A_1|A_2) = 0.9$.

a) Are the events A_1 and A_2 independent? Are the events A_1 and A_2 disjoint? Explain your answers.

What is the probability that the waterworks can pump water on a given day, i.e. what is $P(A_1 \cup A_2)$?

The waterworks was instructed that they in the long run should be able to pump water in 997 out of 1000 days. They decided to install one more pump such that they were able to pump water if at least one of the three pumps were functioning. The new pump shall function independent of the two others. Let A_3 be the event that the new pump is functioning.

b) How large must $P(A_3)$ be in order for the probability to pump water on an arbitrary day shall be 0.997?

Assume the probability that the waterworks can pump water on a day does not depend on theirs ability to pump water the previous days. Let X be the time measured in days until the waterworks is not able to pump water. What is the distribution of X? Explain your answer. What is $\mathrm{E}(X)$ when the third pump is installed?

Problem 2 Non-conformance reports

Charles is responsible for the internal control in a company, and is concerned about the number of reports of non-conformance arriving.

Let N be the number of reports arriving in a period of length t. We assume that reports arrive independent of each other, and that N is Poisson distributed with parameter λt;

$$f(n; \lambda t) = \frac{(\lambda t)^n}{n!} \exp(-\lambda t) \qquad\qquad n = 0, 1, 2, \ldots$$

It is known that $\lambda = 1.5$ messages / week.

a) What is the probability that no non-conformance reports arrives in a period of one week?

What is the probability that more than two non-conformance reports arrive in a period of four weeks?

b) Charles goes on vacation. When he gets back to work three weeks later one report has arrived.

What is the probability that this report has arrived during the first week of his vacation? Explain your answer.

Let T be the time from Charles goes on vacation until this report has arrived. Find the probability distribution for T. Show the derivations, and give reasons for the result.

Charles has the impression that non-conformance is not reported because it takes a lot of time. Therefore he makes a new system for reporting non-conformance. The first year (52 weeks) the new system was used $N = 104$ reports arrived.

Charles want to estimate λ based on these data.

c) Show that the maximum-likelihood estimator for λ is

$$\hat{\lambda}_{SME} = \frac{N}{52}$$

Find the expected value and variance for $\hat{\lambda}_{SME}$. Is $\hat{\lambda}_{SME}$ unbiased? What is the maximum-likelihood estimate for λ?

Problem 3 Contrast fluid

The effect of various types of contrast fluids used in x-ray imaging of hands shall be studied. The contrast fluid is injected into the hand before the x-ray image is shot. The number of images shall be kept low in order to reduce radiation danger - preferably only one image of each hand.

In order to measure the effect one has developed a contrast measure for an image of a hand. Without contrast fluid the measure is denoted K_0 and it varies from person to person, but can be considered as identical for both hands of a person. Previous experience shows that K_0 is normally distributed with expectation μ_0 and standard deviation σ_0. This entails that K_0 is $n(k_0; \mu_0, \sigma_0)$.

a) Assume in this point that $\mu_0 = 25$ and $\sigma_0 = 4$.

Determine the following probabilities:

$P(K_0 \geq 30)$
$P(20 \leq K_0 < 30)$

Now assume that μ_0 and σ_0 are unknown. A study on 10 test persons is used to explore the contrast measure. One x-ray image without using the contrast fluid is taken of one hand for each of the 10 test persons. It results in 10 independent observations of the contrast measure K_0 , see table 1.

Test no. i	1	2	3	4	5	6	7	8	9	10
$k_0(i)$	21	28	19	23	31	32	28	23	28	27

Table 1: Observed contrast without using contrast fluid. It follows that $\bar{k}_0 = 1/10 \sum_{i=1}^{10} k_0(i) = 26$ and $\sum_{i=1}^{10}(k_0(i) - \bar{k}_0)^2 = 166$.

b) Develop the expression for a 90% confidence interval for expected contrast measure μ_0, and determine the actual numerical values.

When using the contrast fluid the contrast in the x-ray images changes according to:

$$K = K_0 + R$$

where R is the effect of the contrast fluid.

Assume that R is normally distributed with expectation μ_R and standard deviation σ_R, ie $n(r; \mu_R, \sigma_R)$. Assume further that K_0 and R have correlation ρ_{0R}, and that also K is normally distributed $n(k; \mu_K, \sigma_K)$.

c) Develop the expression for the expectation μ_K and the standard deviation σ_K to the contrast measure with contrast fluid.

We like to compare contrast measures for two different contrast fluids, type A and type B. Let the effect of each of these be R_A and R_B, and the corresponding contrast measures:

$$K_A = K_0 + R_A$$

$$K_B = K_0 + R_B$$

We assume that all variables are normally distributed, and that R_A and R_B are independent. In order to compare the contrast measures for the two contrast fluids a test is performed: For each type 10 measurements are made. For the 20 test persons the contrast fluid is injected in one hand, an x-ray image is shot, and the contrast measure registered. This results in a set of independent observations of K_A and K_B, see table 2 and 3.

Test no (i)	1	2	3	4	5	6	7	8	9	10
$k_A(i)$	29	38	26	32	40	43	37	31	38	36

Table 2: Observed contrast with contrast fluid type A. It follows that $\bar{k_A} = 1/10 \sum_{i=1}^{10} k_A(i) = 35$.

Test no (i)	1	2	3	4	5	6	7	8	9	10
$k_B(i)$	44	37	46	40	33	29	36	42	35	38

Table 3: Observed contrast with contrast fluid type B. It follows that $\bar{k_B} = 1/10 \sum_{i=1}^{10} k_B(i) = 38$.

Assume in point d) and e) that the standard deviations to K_0 and R are known, $\sigma_0 = 4$ and $\sigma_R = 2$, that the correlation between K_0 and R is known, $\rho_{0R} = 5/16$, and that the standard deviation σ_R and the correlation ρ_{0R} are equal for the two contrast fluids, ie $\mathrm{Var}(R_A) = \mathrm{Var}(R_B) = 2^2$ and $Corr(K_0, R_A) = Corr(K_0, R_B) = 5/16$.

The following hypothesis is forwarded: expected contrast measure for contrast fluid type A and type B are identical. This hypothesis shall be tested against the alternative that the two expectations are different.

d) Test the hypothesis above at a significant level of 0.1 by using the observations in table 2 and 3.

Develop the power for this test for the difference in the expected contrast measures being 2.

An alternative test procedure is to let 10 test persons have contrast fluid type A injected in one hand and type 2 injected in the other hand. Thereafter x-ray images are shot of both hands. This test procedure is performed, and the observations in table 4 are obtained.

Person no (i)	1	2	3	4	5	6	7	8	9	10
$k_A(i)$	29	38	26	32	40	43	37	31	38	36
$k_B(i)$	32	41	28	29	42	41	40	34	42	41

Table 4: Observed contrasts for person no i by using contrast fluid type A, $k_A(i)$, and type B, $k_B(i)$. It follows that $\bar{k}_A = 1/10 \sum_{i=1}^{10} k_A(i) = 35$ and $\bar{k}_B = 1/10 \sum_{i=1}^{10} k_B(i) = 37$.

The same hypothesis as under point d) shall be tested.

e) Explain why this test procedure is better than the other one.

Perform a new test on the hypothesis forwarded in point d) where the favorable properties of this test procedure are used. Use the observations in table 4.

Compare this with the results in point d) and comment.

Calculate the power for this test for difference in expected contrast measure being 2.

Compare the result with the power for the test in point d).

Determine how many test persons are needed in the test procedure in point d) in order to obtain the same power as above.