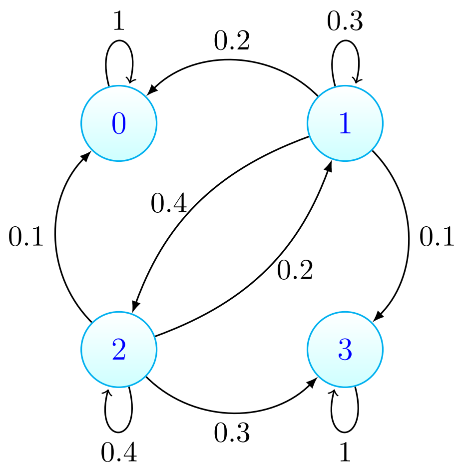
Problem 1

Consider the Markov chain whose transition probability matrix is given by

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.4 & 0.3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

a)

- Draw a state transition diagram and determine the equivalence classes.
  Solution: The state diagram is:



  The state 0 does not communicate with any other state and hence forms one equivalence class. States 1 and 2 communicate and build a second equivalence class, and state 3 forms a third equivalence class as it does not communicate with any other state.
  Thus, there are 3 equivalence classes: $\{0\}$, $\{1, 2\}$, $\{3\}$.

- Which states are recurrent and which states are transient? Justify your answer.
  Solution:
  Recurrence and transiency are class properties, so we will derive the properties on class level. Class $\{0\}$ is recurrent, since the probability starting in 0 to re-enter state 0 is one. The same is true for $\{3\}$, which means that it is also recurrent. However, class $\{1, 2\}$ is transient, since it is not possible to re-enter the class once it is left.

- Calculate the following probabilities:

$$P(X_4 = 3 \mid X_3 = 1, X_2 = 2) \quad \text{and} \quad P(X_2 = 3 \mid X_0 = 1)$$

Solution:
We have

$$P(X_4 = 3 \mid X_3 = 1, X_2 = 2) = P(X_4 = 3 \mid X_3 = 1) = P(X_1 = 3 \mid X_0 = 1) = 0.1$$

Further

$$P^2 = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.30 & 0.17 & 0.28 & 0.25 \\ 0.18 & 0.14 & 0.24 & 0.44 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

so that $P(X_2 = 3 \mid X_0 = 1)$ follows as $P^2[2, 4] = 0.25$. Alternatively,

$$P(X_2 = 3 \mid X_0 = 1) = 0.3 \cdot 0.1 + 0.4 \cdot 0.3 + 0.1 \cdot 1 = 0.25.$$

b)

- Compute the probability of absorption into state 0 starting from state 1.

Solution:
This can be solved using a first step analysis. Let A denote that the chain reaches state 0. Define

$$u_i = P(A \mid X_0 = i), \quad i = 0, 1, 2, 3$$

Obviously $u_0 = 1$ and $u_3 = 0$. Further

$$\begin{aligned}
u_1 &= 0.2 + 0.3u_1 + 0.4u_2 \Rightarrow 0.7u_1 = 0.2 + 0.4u_2 \Rightarrow u_1 = \frac{2}{7} + \frac{4}{7}u_2 \\
u_2 &= 0.1 + 0.2u_1 + 0.4u_2 \\
\Rightarrow u_2 &= \frac{1}{10} + \frac{2}{10} \cdot (\frac{2}{7} + \frac{4}{7}u_2) + \frac{4}{10}u_2 \\
\Rightarrow \frac{34}{70}u_2 &= \frac{11}{70} \\
\Rightarrow u_2 &= \frac{11}{34} \\
\Rightarrow u_1 &= \frac{2}{7} + \frac{22}{119} \approx 0.47
\end{aligned}$$

The probability of absorption into state 0 starting from state 1 is given by $u_1 = 0.47$.

- Starting in state 1, compute the expected time spent in each of states 1 and 2 prior to absorption in state 0 or 3.

  Solution:

  Let

  $$P_T = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.3 & 0.4 \\ 0.2 & 0.4 \end{pmatrix} \end{array},$$

  be the matrix specifying only the transition probabilities from transient into transient states. To find the desired quantities we need to compute

  $$\mathbf{S} = (\mathbf{I}-\mathbf{P}_T)^{-1} = \begin{pmatrix} \frac{7}{10} & -\frac{4}{10} \\ -\frac{2}{10} & \frac{6}{10} \end{pmatrix}^{-1} = \frac{50}{17}\begin{pmatrix} \frac{6}{10} & \frac{4}{10} \\ \frac{2}{10} & \frac{7}{10} \end{pmatrix} = \begin{pmatrix} \frac{30}{17} & \frac{20}{17} \\ \frac{10}{17} & \frac{35}{17} \end{pmatrix} \approx \begin{pmatrix} 1.765 & 1.176 \\ 0.588 & 2.059 \end{pmatrix}$$

  The desired quantities are in $S_{1,1}$ and $S_{1,2}$. Thus, starting in state 1, the expected number of periods spent in state 1 prior to absorption is 1.765, whereas the expected number of periods spent in state 2 is 1.176.

Problem 2

Let $\{X_n, n = 0, 1, \ldots\}$ denote a branching process in which all individuals are assumed to have offsprings independently of each other. X_n denotes the population size at the n-th generation and we assume that $X_0 = 1$. By the end of its life time, each individual is assumed to have produced no offspring with probability $P_0 = \frac{1}{8}$, one offspring with probability $P_1 = \frac{1}{2}$ and two offspring with probability $P_2 = \frac{3}{8}$

a)
- Explain why this process is a Markov chain.

  Solution (2 point):

  As all individuals have offsprings independently of others, the population size of the n-th generation will only depend on the population size of the $(n-1)$-th generation and the given chain is then a Markov chain.

- Derive the state space. Which states are transient and which states are recurrent?

  Solution (2 point):

  This chain is defined on the state space $S = \{0, 1, 2, ...\}$ in which state 0 is recurrent while all the other states are transient.

b) Compute the expected number of offsprings of a single individual. What is the probability that the population will die out?

Solution (4 points):

The expected number of offsprings for each individual is

$$\mu = \sum_{j=0}^{2} jP_j = 0 \cdot \frac{1}{8} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{3}{8} = \frac{10}{8} > 1$$

The probability that the population will die out π_0 is then given as the smallest solution to the equation

$$\pi_0 = \sum_{j=0}^{2} \pi_0^j P_j = \frac{1}{8} + \pi_0 \cdot \frac{1}{2} + \pi_0^2 \cdot \frac{3}{8}.$$

This gives

$$\begin{aligned} \frac{1}{8} - \pi_0 \cdot \frac{1}{2} + \pi_0^2 \cdot \frac{3}{8} &= 0 \\ &\Downarrow \\ \pi_0 &= \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{3}{16}}}{2 \cdot \frac{3}{8}} \\ &= \frac{\frac{1}{2} \pm \frac{1}{4}}{\frac{3}{4}} \Rightarrow \pi_0 = 1, \pi_0 = \frac{1}{3}. \end{aligned}$$

The probability that the population will die out it $\pi_0 = \frac{1}{3}$.

Problem 3

An insurance company pays out claims on its life insurance policies in accordance with a Poisson process having rate $\lambda = 6$ claims per week. Let $N(t)$ be the number of insurance claims at time t (measured in weeks) and assume that $N(0) = 0$.

a)
- What is the expected time until the fifteenth insurance claim is paid?
  Solution:
  The expected time until the fifteenth insurance claim is paid is $E(S_{15}) = \frac{15}{\lambda} = \frac{15}{6} = 2.5$ weeks.
- Find $E(N(4) - N(2) \mid N(1) = 5)$
  Solution:
  Independent increments give
  $E(N(4) - N(2) \mid N(1) = 5) = E(N(4) - N(2)) = E(N(2)) = 2\lambda = 12$ claims.

- Compute also $P(N(3) \geq 12)$.
  Solution:
  $P(N(3) \geq 12) = 1 - P(N(3) < 12) \overset{\text{Table}}{=} 1 - 0.0549 = 0.9451$

Assume that the amount of money paid on each policy are independent exponentially distributed random variables with common mean 12000 kroner. Assume also that the amount of money paid on each policy is independent of the number of claims that are paid out.

b) What is the mean and variance of the total amount of money paid by the insurance company in a four-week span?

Solution:

Let N denote the number of insurance claims per week and X_i the amount of money paid on the $i-th$ policy. We know that $E(N) = Var(N) = \lambda = 6$. Then

$$E(\sum_{i=1}^{N} X_i) = E(E(\sum_{i=1}^{N} X_i \mid N)) = \ldots = E(N) \cdot E(X).$$

Here, the expected total amount of money paid during a four-week time span is $4 \times 6 \times 12000 = 288000$ kroner.

The variance for a one-week period can be derived similarly

$$\begin{aligned} Var(\sum_{i=1}^{N} X_i) &= Var(E(\sum_{i=1}^{N} X_i \mid N)) + E(Var(\sum_{i=1}^{N} X_i \mid N)) \\ &= \ldots = E(N)Var(X) + Var(N)E(X)^2 \\ &= 6 \cdot 12000^2 + 6 \cdot 12000^2 = 1.728 \times 10^9 \end{aligned}$$

Thus for a four-week span the variance is 6.912×10^9 kroner^2.

Problem 4

Biathlon commonly refers to the winter sport that combines cross-country skiing and rifle shooting. Assume that the inhabitants of Oslo want to improve their Biathlon skills and go to a popular skiing area to train. There is a stadium with three public shooting stands available. Skiers arrive at the shooting stands according to a Poisson process with rate 5 skiers per minute, i.e. $\lambda = 1/12$ skier per second. If a shooting stand is free an entering skier immediately starts to shoot and then leaves directly the stadium. If all stands are occupied he waits in line and then goes to the first free shooting stand. The time a skier spends at either of the shooting stands is independent of the other skiers and exponentially distributed with mean 30 seconds, i.e. with rate $\mu = 1/30$.

Let $X(t)$ denote the number of skiers in the stadium at time t, i.e. skiers who are either shooting or waiting in line until a shooting stand becomes free. We assume that $X(0) = 0$.

a)
- Explain briefly why $X(t)$ is a birth-death process and give all birth and death rates.

  Solution: (4 points)

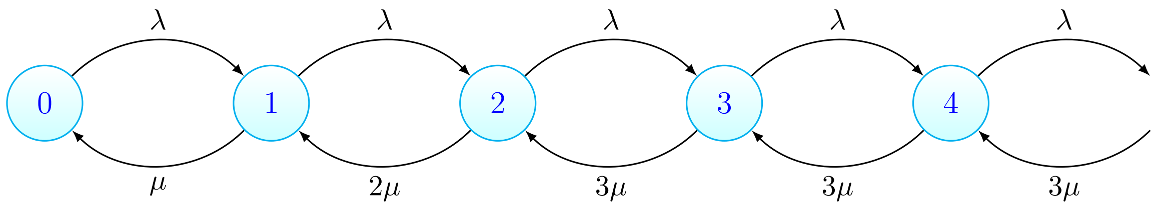
  The given process is a birth and death process as the number of skiers in the stadium either increase with one (birth) or decrease with one (death) and all times until the next arrival (birth) and termination of shooting (death) are independent and exponentially distributed. The birth rates are given by

  $$\lambda_n = \lambda, n = 0, 1, 2, \ldots$$

  while the death rates are

  $$\mu_1 = \mu, \mu_2 = 2\mu, \mu_n = 3\mu, \quad \text{for } n = 3, 4, \ldots$$

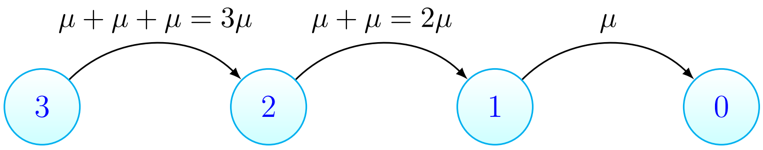
  This is a M/M/3 queue and the transition diagram is



- If $X(t) = 3$, what is the expected time until all these three skiers have finished shooting.

  Solution: (4 points)

  The time until a skier is finished with shooting is exponentially distributed with rate $\mu = 1/30$. The time until the next skier is finished is then the minimum of independent and exponentially distributed variables which is exponential with rate equal to the sum of the individual rates. Meaning that eather of the three completions, assuming $X(t) = 3$ will move the system to $X(t) = 2$, and so on.



The expected time W until all of the three skiers are finished is

$$E(W) = \frac{1}{3\mu} + \frac{1}{2\mu} + \frac{1}{\mu} = \frac{30}{3} + \frac{30}{2} + \frac{30}{1} = \frac{330}{6} \approx 55 \text{ seconds.}$$

b) Starting at time 0, what is the expected time until $X(t) = 3$ for the first time.

Solution: (4 points)

Let T_i denote the time, starting from start i, it takes for the process to enter state $i+1$, $i \geq 0$. Hence the expected time to reach state 3 (in seconds) is:

$$\begin{aligned}
E(T_0) &= \frac{1}{\lambda_0} = \frac{1}{\lambda} = 12 \\
E(T_1) &= \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1} E(T_0) = 12 + \frac{12}{30} \cdot 12 = 16.8 \\
E(T_2) &= \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2} E(T_1) = 12 + \frac{2 \cdot 12}{30} \cdot 16.8 = 25.44 \\
&\Rightarrow \\
\sum_{i=0}^{2} E(T_i) &= 12 + 16.8 + 25.44 = 54.24 \text{ seconds.}
\end{aligned}$$

In the remaining questions, first express the answers as functions of λ and μ. Thereafter, compute the numerical answer for the parameter values given.

c)
- Derive the limiting probabilities for $X(t)$.

  Solution:

  In general

  $$P_n = \frac{\theta_n}{\sum_{n=0}^{\infty} \theta_n}$$

  where $\theta_0 = 1$ and

  $$\theta_n = \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n}$$

Here,

$$\begin{aligned}
\theta_1 &= \frac{\lambda_0}{\mu_1} = \frac{\lambda}{\mu} \\
\theta_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = \frac{\lambda^2}{2\mu^2} \dots \\
\theta_n = \theta_n &= \frac{\lambda_0 \cdots \lambda_{n-1}}{\mu_1 \cdots \mu_n} = \frac{\lambda^n}{2 \cdot 3^{n-2}\mu^n}, \quad \text{for } n > 2
\end{aligned}$$

With $\lambda = 1/12$ and $\mu = 1/30$ we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \theta_n &= 1 + \frac{\lambda}{\mu} + \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{3^{n-2}} \\
&= 1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 \sum_{n=0}^{\infty}\left(\frac{\lambda}{3\mu}\right)^n \\
&\overset{\text{geom. series}}{=} 1 + \frac{\lambda}{\mu} + \frac{1}{2}\left(\frac{\lambda}{\mu}\right)^2 \frac{1}{1-\frac{\lambda}{3\mu}} \\
&= 1 + 2.5 + \frac{2.5^2}{2\cdot(1-\frac{2.5}{3})} = \frac{89}{4} = 22.25
\end{aligned}$$

Hence, the limiting probabilities result as:

$$\begin{aligned}
P_0 &= \frac{1}{\sum_{n=0}^{\infty}\theta_n} = \frac{4}{89} \approx 0.045 \\
P_1 &= \theta_1 P_0 = \frac{30}{12}\frac{4}{89} = \frac{10}{89} \approx 0.112 \\
P_n &= \theta_n P_0 = \frac{\lambda^n}{2\cdot 3^{n-2}\mu^n}\frac{4}{89} = \left(\frac{5}{2}\right)^n \frac{1}{3^{n-2}}\frac{2}{89} = \frac{25}{4}\left(\frac{5}{6}\right)^{n-2}\frac{2}{89} = \frac{25}{178}\left(\frac{5}{6}\right)^{n-2}, \qquad n \geq 2
\end{aligned}$$

- In the long run what proportion of skiers can start shooting immediately after arrival (i.e. without first waiting until a shooting stand becomes free)?

  Solution:

  A skier can start shooting without waiting if either no, only one or only two shooting stands are busy. Hence the desired probability is given by

$$P_0 + P_1 + P_2 = \frac{8+20+25}{178} = \frac{53}{178} \approx 0.30$$

d)

- Compute the expected number of skiers in the stadium after a long time has passed.

  Solution:

  Let L be the expected number of skiers in the stadium. Then

$$\begin{aligned}
L &= \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\theta_n P_0 \\
&= \sum_{n=0}^{\infty} n \left(\frac{\lambda^n}{2 \cdot 3^{n-2} \cdot \mu^n} - \frac{\lambda}{2\mu} \right) \cdot \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{1-\lambda/(3\mu)}} \\
&= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{1-\lambda/(3\mu)}} \left[\sum_{n=0}^{\infty} n \frac{1}{2 \cdot 3^{n-2}} \left(\frac{\lambda}{\mu}\right)^n - \frac{\lambda}{2\mu} \right] \\
&= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{1-\lambda/(3\mu)}} \left[\frac{3^2}{2} \sum_{n=0}^{\infty} n \left(\frac{\lambda}{3\mu}\right)^n - \frac{\lambda}{2\mu} \right] \\
&= \frac{1}{1 + \frac{\lambda}{\mu} + \frac{1}{2} \cdot \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{1-\lambda/(3\mu)}} \left[\frac{3\lambda/\mu}{2(1 - \lambda/(3\mu))^2} - \frac{\lambda}{2\mu} \right] \\
&\approx 6.011236
\end{aligned}$$

  The expected number of skiers in the stadium after a long time is 6.011.

- Use Little's formula to find the average amount of time each skier spends in the stadium.

  Solution:

  From Little's form we get that the average amount of time W a skier spents in the system is given by

$$\begin{aligned}
W &= \frac{L}{\lambda} \\
&= \frac{6.011236}{1/12} \approx 72.13483
\end{aligned}$$

  A skier spends on average 72.13 seconds in the stadium.