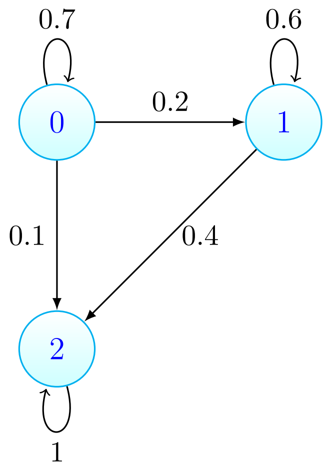
Problem 1

a) The transition matrix is

$$P = \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \end{pmatrix}$$

The state diagram is:



No states communicate, so that we have three equivalence classes. State 0 and state 1 are transient, since it is not possible to re-enter the states once they are left, state 2 is recurrent since the probability starting in 2 to re-enter state 2 is one.

b) – $P(X_3 \neq 2, X_1 \neq 1 | X_0 = 0) = 0.7 \cdot 0.7 \cdot 0.7 + 0.7 \cdot 0.2 \cdot 0.6 + 0.7 \cdot 0.7 \cdot 0.2 = 0.525$

– $P(X_4 = 2 | X_2 = 0, X_0 = 0) = P(X_4 = 2 | X_2 = 0) = P(X_2 = 2 | X_0 = 0) = 0.25$

– Let A be the event that the machine is never in state 1. We compute $P(A)$ using a first-step analysis:

$$\begin{aligned} P(A) &= P(A|X_1 = 0)P(X_1 = 0) + P(A|X_1 = 1)P(X_1 = 1) + P(A|X_1 = 2)P(X_1 = 2) \\ &= 0.7 \cdot P(A) + 0.1 \\ &\Rightarrow \end{aligned}$$

$$0.3P(A) = 0.1 \Rightarrow P(A) = \frac{1}{3}$$

c) $E(T)$ can either be found straightforwardly using a first step analysis. Alternatively, we can use the transition matrix P_T which only includes the transition probabilities for the transient states and compute $S = (I - P_T)^{-1}$. Here, the entries s_{ij} denote the expected time spent in state j given start in state i.

First step analysis: Let $m_i = E(T|X_0 = i)$

$$
\begin{aligned}
m_0 &= 1 + 0.7m_0 + 0.2m_1 + 0.1 \cdot 0 \\
m_1 &= 1 + 0.6m_1 + 0.4 \cdot 0,
\end{aligned}
$$

so that

$$
\begin{aligned}
0.4m_1 &= 1 \to m_1 = 10/4 \\
0.3m_0 &= 1 + 0.2 \cdot \frac{10}{4} \to m_0 = 5
\end{aligned}
$$

The remaining number of weeks the machine will work is 5 weeks.

Alternatively, we could compute this as:

$$
\begin{aligned}
S = (I - P_T)^{-1} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.7 & 0.2 \\ 0 & 0.6 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 0.3 & -0.2 \\ 0 & 0.4 \end{pmatrix}^{-1} \\
&= \frac{1}{0.12} \begin{pmatrix} 0.4 & 0.2 \\ 0 & 0.3 \end{pmatrix} = \begin{pmatrix} \frac{10}{3} & \frac{5}{3} \\ 0 & \frac{5}{2} \end{pmatrix}
\end{aligned}
$$

which gives the same result $\frac{10}{3} + \frac{5}{3} = 5$.

Problem 2

a)
   - $E(T_2) = \frac{1}{3}$ hours, i.e. 20 minutes.

   - $P(N(1) = 3) = \frac{3^3}{3!} \exp(-3) = 0.2240418$

   - $E(S_{14}|N(3) = 8) = 3 + \frac{6}{3} = 5$ hours.

b) The office is supposed to open at 8AM. What is the distribution of the amount of time the clerk Oscar has to wait until his first customer arrives?

   $P(T_1 > \tau) = P(N(\tau) = 0) = \exp(-\lambda\tau)$. Thus $T_1 \sim$ exponential(3).

   Assume, Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period?

   $P(T_1 > 2h) = 1 - P(T_1 \leq 2h) = 1 - (1 - \exp(-\frac{3}{h} \cdot 2h)) = \exp(-6) = 0.002478752$

c) Let N be the number of customers who arrive before Oscar is at his job and let T denote his arrival time. Then

$$E(N) = \int_0^{3/2} E(N|T = \tau)p_T(\tau)d\tau = \int_0^{3/2} 3\tau \cdot \frac{2}{3} d\tau = 3\frac{1}{2} \cdot \frac{2}{3}\tau^2|_0^{3/2} = \frac{9}{4} = 2.25.$$

Problem 3

a) X(t) is a birth and death process as X(t) denotes the number of individuals in the system at time t. The waiting time until another customer arrives (birth) and the waiting time until a customer leaves (death) are independent and exponentially distributed, and no more than one event (one arrival, one departure) can happen at the exact same time. That means the number of customers either increase with one (birth) or decrease with one (death). The birth rates are given by

$$\lambda_n = \lambda\alpha_n \quad \text{for } n = 0, 1, 2, 3, 4$$
$$\mu_n = \mu \quad \text{for } n = 1, 2, 3, 4$$

and $\mu_0 = 0$.

b) Let T_i denote the time, starting from start i, it takes for the process to enter state $i+1$, $i \geq 0$. Hence the expected time to reach state 2 (in minutes) is:

$$E(T_0) = \frac{1}{\lambda_0} = \frac{1}{\lambda} = \frac{1}{20} = 3 \text{ minutes}$$
$$E(T_1) = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1}E(T_0) = \frac{1}{\frac{3}{4}\lambda} + \frac{\mu}{\frac{3}{4}\lambda} \cdot \frac{1}{\lambda} = \frac{1}{\frac{3}{4}20} + \frac{20}{\frac{3}{4}20} \cdot \frac{1}{20} = \frac{8}{3 \cdot 20} = 8 \text{ minutes}$$
$$\Rightarrow$$

$E(T_0) + E(T_1) = 3 + 8 = 11$ minutes.

Let $X_1 \sim \exp(\mu)$ denote the service time of the first customer (note after arrival the first customer is not in line but immediately served) and let $X_2 \sim \exp(\frac{3}{4}\lambda)$ be the inter-arrival time between customer 1 and 2. As both variables are independent and exponentially distributed:

$$P(X_1 < X_2) = \frac{\mu}{\mu + \frac{3}{4}\lambda} = \frac{20}{20 + \frac{3}{4} \cdot 20} = \frac{20}{35} = \frac{4}{7} \approx 0.57.$$

c) In general, the limiting probabilities for a birth and death process are given by

$$P_k = \frac{\theta_k}{\sum_{n=0}^{\infty} \theta_n}$$

with $\theta_0 = 1$. Here

$$\begin{aligned}
\theta_1 &= \frac{\lambda_0}{\mu_1} = \frac{\lambda}{\mu} \\
\theta_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2} = \frac{3}{4}\left(\frac{\lambda}{\mu}\right)^2 \\
\theta_3 &= \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} = \frac{3}{8}\left(\frac{\lambda}{\mu}\right)^3 \\
\theta_4 &= \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{\mu_1\mu_2\mu_3\mu_4} = \frac{3}{32}\left(\frac{\lambda}{\mu}\right)^4
\end{aligned}$$

Thus

$$\sum_{n=0}^{4}\theta_n = 1 + 1 + \frac{3}{4} + \frac{3}{8} + \frac{3}{32} = \frac{32+32+24+12+3}{32} = \frac{103}{32}$$

and

$$P_0 = \frac{32}{103} \qquad P_1 = \frac{32}{103} \qquad P_2 = \frac{24}{103} \qquad P_3 = \frac{12}{103} \qquad P_4 = \frac{3}{103}$$

d) In the long run:

- What is the probability that an arriving customer will join the queueing system and not leave immediately?

$$\begin{aligned}
P(\text{do not leave}) &= \sum_{k=0}^{4} P(\text{do not leave}|X(t)=k)P_k \\
&= \frac{32}{103} + \frac{3}{4}\cdot\frac{32}{103} + \frac{1}{2}\cdot\frac{24}{103} + \frac{1}{4}\cdot\frac{12}{103} + 0\cdot\frac{3}{103} = \frac{32+24+12+3}{103} = \frac{71}{103} \approx 0.69.
\end{aligned}$$

- What is the expected number of customers in the queueing system.

$$L = 0\cdot P_0 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + 4\cdot P_4 = \frac{32+48+36+12}{103} = \frac{128}{103} \approx 1.24$$

- Use Little's formula to compute the expected time a customer (which decides to join the system) will totally spend in the system?

  Little's formula is $L = \lambda \cdot W$. Note λ is here not the same as in the problem description. We know that

$$\begin{aligned}
L &= \frac{128}{103} \\
\lambda &= 20\cdot\frac{71}{103}
\end{aligned}$$

Thus

$$W = \frac{\frac{128}{103}}{20 \cdot \frac{71}{103}} \approx 0.09014085 \text{ hours}$$

That means 5.41 minutes.