Norges teknisk–
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English

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EXAM IN COURSE TMA4265 STOCHASTIC PROCESSES

Wednesday 7. August, 2013
Time: 9:00–13:00

Permitted aid items:

- Yellow A-5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences)
- *Tabeller og formler i statistikk*, Tapir Forlag
- K. Rottmann: *Matematisk formelsamling*
- Calculator HP30S, Citizen SR-270X, Citizen SR-270X College

The results from the exam are due by 28. august, 2013

Problem 1 THE GARAGE

In a garage N cars each have their own permanent parking place. We assume that the car drivers behave independently of each other. Each of the car drivers that has not yet parked their car, will during a short time interval of length h park the car in its own spot with probability $\lambda h + o(h)$, while each of the parked cars during a time interval of length h will leave its parking space with probability $\mu h + o(h)$ $(\lambda, \mu > 0)$. Let $X(t)$ be the number of parking spaces in the garage that are occupied at time t. You may assume that $X(t)$ is a Markov process with stationary transition probabilities $P_{ij}(t) = P\big(X(s+t) = j \mid X(s) = i\big)$.

a) Explain briefly why

$$P_{i,i+1}(h) = (N-i)\lambda h + o(h)\,;\;\; i = 0,1,\ldots,N-1$$

$$P_{i,i-1}(h) = i\mu h + o(h)\,;\;\; i = 1,2,\ldots,N$$

$$P_{ii}(h) = 1 - \big[(N-i)\lambda + i\mu\big]h + o(h)\,;\;\; i = 0,1,\ldots,N$$

Using the equations in point a) one can derive the Kolmogorov forward differential equations. It can be shown that these equations are given as follows:

$$P'_{i0}(t) = -\lambda N P_{i0}(t) + \mu P_{i1}(t)$$

$$P'_{ij}(t) = (N-j+1)\lambda P_{i,j-1}(t) - [(N-j)\lambda + j\mu]P_{ij}(t) + (j+1)\mu P_{i,j+1}(t)\,,\;\; 1 \le j \le N-1$$

$$P'_{iN}(t) = \lambda P_{i,N-1}(t) - N\mu P_{iN}(t)$$

b) Let $\pi_j = \lim_{t\to\infty} P_{ij}(t)$ denote the probability that there are j cars in the garage in the long run.

Determine the limiting distribution $(\pi_0, \pi_1, \ldots, \pi_N)$. (Hint: $\sum_{j=0}^{N} \binom{N}{j} x^j = (1+x)^N$.)

c) Assume that $X(0) = i$ and let $M(t) = \mathrm{E}[X(t)]$ be the expected number of cars in the garage at time t. Show that $M(t)$ satisfies the differential equation

$$M'(t) = N\lambda - (\lambda+\mu)M(t).$$

Determine $M(t)$. Find $\lim_{t\to\infty} M(t)$.

(Hint: The law of double expectation can be useful.)

Problem 2 THE AIRPORT

We assume in this problem that the arrival of (single) passengers at the self service terminals of an airline company (at a given airport) corresponds to the events of a Poisson process $N(t)$ with parameter $\lambda > 0$ and with starting time $t = 0$. We let $N(s,t] = N(t) - N(s)$ denote the number of arrivals during the time interval $(s,t]$ for all $0 \le s < t$.

It is assumed that all passengers that arrive, immediately finds an available terminal. This is done in the model by assuming an infinity of terminals.

Let $Y(t)$ be the number of busy terminals at time $t \ge 0$.

In points a) and b) it is assumed that all passengers spend an equal amount of time, a minutes, at the terminals. Hence, a is a fixed positive constant.

a) Explain why $Y(t) = N(t-a, t]$ for all $t \geq a$.

Establish the probability distribution for $Y(t)$ for a given point in time $t \geq a$.

In addition, find the probability distribution for $Y(t)$ for $0 < t < a$.

b) Find, for all $t > s > a$, the probability that none of the terminals are busy both at time s and at time t, that is,

$$P(Y(s) = 0 \cap Y(t) = 0).$$

In practice it is unrealistic to assume that the service time is a constant a. Therefore, we shall for the rest of this problem assume that the service time is a stochastic variable A with distribution $P(A \leq a) = G(a)$ and finite expectation m_A. We also assume that the service times for different passengers are stochastically independent and independent of the arrival process.

c) Use Little's formula to find the average number of busy terminals when the system is in equilibrium. Check that the answer in point a) agrees with this result.

Since the arrival process is a Poisson process, it is possible to calculate the exact distribution of $Y(t)$. This will be the task of the next point.

d) Show that $Y(t)$ for all $t \geq 0$ is Poisson distributed with parameter $\lambda \int_0^t (1 - G(s))ds$.

Check that the answer is in agreement with the result of the previous point.

Hint: You may e.g. use that

$$P(Y(t) = k) = \sum_{n=k}^{\infty} P(Y(t) = k | N(t) = n) P(N(t) = n)$$

and assume as known that if $N(t) = n$ is given, the n arrivals will be independent and uniformly distributed in the interval $(0, t]$. This leads to the result that $Y(t)$, given $N(t) = n$, is binomially distributed.

Problem 3 BROWNIAN MOTION

Let $\{B(t), t \geq 0\}$ denote a standard Brownian motion, that is, $\sigma = 1$.

a) Write down the definition of this process.

b) Find $P(B(2) \geq 3)$.

Given that $B(1) = 1$, what is the probability that $B(2) \geq 3$?

c) Let T_3 denote the first time when the process reaches the value 3.

Find $P(T_3 \leq 2)$. You are expected to derive the result. It is not sufficient to merely use a formula from the textbook.

Hint: Start with $P(B(2) \geq 3)$.

Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$
$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{og} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$\begin{aligned} P_{i,i+1}(h) &= \lambda_i h + o(h) \\ P_{i,i-1}(h) &= \mu_i h + o(h) \\ P_{i,i}(h) &= 1 - (\lambda_i + \mu_i)h + o(h) \\ P_{ij}(h) &= o(h) \quad \text{for } |j - i| \ge 2 \end{aligned}$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \ge 0$ are birth rates, $\mu_i \ge 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i\,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}\,,\ \ i \ne j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{og} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{og} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{og} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[S W_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad ,$$