NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Processes**

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Examination date: August 2014

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Calculator HP30S, CITIZEN SR-270X or CITIZEN SR-270X College, Casio fx-82ES PLUS with empty memory.
- Statistiske tabeller og formler, Tapir.
- K. Rottmann: Matematisk formelsamling.
- One yellow, stamped A5 sheet with own handwritten notes.

Other information:

- All answers must be justified.
- In your solution you can use English and/or Norwegian.

Language: English

Number of pages: 6

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

A machine is inspected weekly in order to determine its condition. The condition of the machine can be one of the following three states:

- 0: perfect
- 1: reasonable
- 2: broken

A machine in perfect condition is still perfect after one week with probability 0.7, with probability 0.2 the state changes to reasonable. A machine in reasonable condition is still reasonable after one week with probability 0.6, and broken with probability 0.4. A broken machine is not working and cannot be repaired. It stays broken.

a)
- Formulate the transition matrix of a Markov chain that describes the state of the machine, and draw the corresponding transition diagram.
- Determine the equivalence classes and decide which states are recurrent and which states are transient? Justify your answer.

b) We start to observe a new machine this week, so that $X_0 = 0$. Calculate:

- $P(X_3 \neq 2, X_1 \neq 1 | X_0 = 0)$
- $P(X_4 = 2 | X_2 = 0, X_0 = 0)$
- The probability that the machine will never be in state 1.

c) Assume again that we start to observe the state of the machine this week, in which $X_0 = 0$, and define a random variable

$$T = \min\{n \geq 1 : X_n = 2\}.$$

Describe two different strategies that can be used to find the expectation $E(T)$.

Use ONE of these strategies to calculate the value of $E(T)$.

Problem 2

Customers arrive at a shipping office according to a Poisson process with rate $\lambda = 3$ per hour. Let $N(t)$ denote the number of customers at time t. Let T_i denote the time between the arrival of the $i-1$-th and i-th customer and let $S_n = \sum_{i=1}^{n} T_i$.

a) Find

- $E(T_2)$
- $P(N(1) = 3)$
- $E(S_{14}|N(3) = 8)$

b) The office opens at 8AM. What is the distribution of the amount of time the clerk Oscar has to wait until his first customer arrives?

Assume, Oscar overslept and came in at 10AM. What is the probability that no customers came in the two-hour period?

c) The office opens at 8AM. Assume that the arrival time of Oscar is uniformly distributed between 8AM and 9:30AM. What is the expected number of customers who arrive before Oscar is at his job?

Problem 3

The restaurant chain "Shake Shack" is said to have excellent hamburgers in New York City. There is one small stand in Madison Square Park. Assume there is one server who prepares the food fresh for the customers. Customers arrive at the queueing system according to a Poisson process with intensity $\lambda = 20$ (hour^{-1}). Suppose that a customer that finds n people in the queueing system upon its arrival will only join the queueing system with probability

$$\alpha_n = \frac{4-n}{4}, \qquad n = 0, 1, 2, 3, 4$$

Customers that join the system are served in order of arrival and the service times are assumed to be independent and exponentially distributed with a mean service time equal to 3 minutes, i.e. rate $\mu = 20$ (hour^{-1}). Further, we assume that the customer's service time is independent of the arrival process.

Let $X(t)$ denote the number of customers in the system (including the one that might be under service) at time point t. Assume that $X(0) = 0$.

a) Explain why $X(t)$ is a birth and death process and give the birth and death rates.

In the remaining questions, first express the answers as functions of λ and μ. Thereafter, compute the numerical answer for the parameter values given.

b) Starting at time 0, what is the expected time until $X(t) = 2$ for the first time?

What is the probability that the first customer has departed the queueing system before the next customer arrives?

c) Derive the limiting probabilities for $X(t)$ and show that these are equal to

$$P_0 = \frac{32}{103} \qquad P_1 = \frac{32}{103} \qquad P_2 = \frac{24}{103} \qquad P_3 = \frac{12}{103} \qquad P_4 = \frac{3}{103}$$

d) In the long run:

- What is the probability that an arriving customer will join the queueing system and not leave immediately?
- What is the expected number of customers in the queueing system.
- Use Little's formula to compute the expected time a customer (who decides to join the system) will totally spend in the queueing system?

Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1, S_2, \ldots, S_n | N(t)}(s_1, s_2, \ldots, s_n | n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$
$$P_{i,i-1}(h) = \mu_i h + o(h)$$
$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$
$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \, , \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \, , \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$