Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

English

Contact during exam:
Arvid Næss 73 59 70 53/ 99 53 83 50

EXAM IN COURSE TMA4265 STOCHASTIC PROCESSES

Wednesday, December 19, 2012
Time: 9:00–13:00

Permitted aid items:

- Yellow A-5 sheet with your own handwritten notes (stamped by the Department of Mathematical Sciences)
- *Tabeller og formler i statistikk*, Tapir Forlag
- K. Rottmann: *Matematisk formelsamling*
- Calculator HP30S, Citizen SR-270X, Citizen SR-270X College

The results from the exam are due by January 23, 2013.

Problem 1 - A Repair System

Consider a system that has three components which must all function for the system to function. If a component fails it is replaced by a component from a replacement storage. After the failed component has been replaced and the system is again functioning, the failed component is repaired and then placed in the replacement storage. It is assumed that the replacement storage can only contain one component. It is also assumed that components can only fail when in function. Note that the system is in function only when three components are functioning.

For the modelling we simplify and let the possible states of the system be characterized as follows:

- State 3 : Three components are in function, one is in the replacement storage
- State 2 : Two components are in function, replacement with component from storage is ongoing.
- State 1 : Three components are in function, none is in the replacement storage
- State 0 : Two components are in function, none is in the replacement storage

$X(t)$ denotes the state of the system at time t. It is assumed that $X(t)$ is a Markov chain in continuous time with stationary transition probabilities.

For calculating the transition rates q_{ij} (cf. formula sheet), it is assumed that:

A component which is in function at time t fails during the time interval $(t, t+h]$ with probability $\lambda h + o(h)$, independent of the other components.

Repair work on a component at time t is finished during the time interval $(t, t+h]$ with probability $\mu h + o(h)$, independent of the condition of the other components.

The expected time to replace a component which fails with a component from the replacement storage is $1/\gamma$.

a) Show that the transition rates for the Markov chain are as given in the communication diagram below.

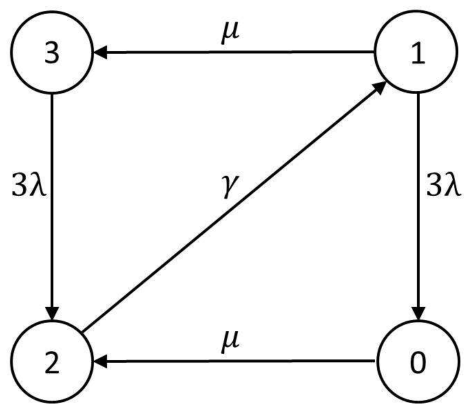

Figure 1:

b) Establish a set of equations that determines the stationary distribution $\boldsymbol{p} = (p_0, p_1, p_2, p_3)$ for the process. (You are not asked to solve the equations.)

c) Given that the system does not function at time t, where t is large, determine (approximately) the probability that the system is in function at time $t+h$ expressed in terms of the p_i, $i = 0, \ldots, 3$, and parameters of the system. (h is a small number.)

d) Assume that $X(0) = 3$. Find the probability that the process after having left state 3 will visit state 0 before it returns to state 3. Based on this probability, justify that if $\lambda \ll \mu$ ($\ll$ = much less than) the stationary distribution in point b) can be approximated by putting the transition intensity $q_{10} = 0$. Determine the approximation to $\boldsymbol{p}$ that is obtained in this way. (Hint: Simplify the communication diagramme.)

Problem 2 - A Queueing System

Consider an $M/G/1$ queueing system in which the first customer in a busy period has the service time distribution G_1 (and service time S_1) and all others have service time distribution G_2 (and service time S_2). Let S denote the service time of a customer chosen at random.

a) Argue why $a_0 = P_0 = 1 - \lambda \mathrm{E}[S]$. Explain the symbols used.

b) Determine $\mathrm{E}[S]$ in terms of a_0, $\mathrm{E}[S_1]$ and $\mathrm{E}[S_2]$.

c) Use the results in the previous point to show that

$$\mathrm{E}[B] = \frac{\mathrm{E}[S_1]}{1 - \lambda \mathrm{E}[S_2]},$$

where B denotes the length of a busy period. (Hint: Express P_0 in terms of $\mathrm{E}[B]$ and $\mathrm{E}[I]$ (I = idle period), and remember that $\mathrm{E}[I] = 1/\lambda$.)

Problem 3 - Brownian Motion

A stochastic process $B(t)$ ($t \geq 0$) is a standard Brownian motion process if: (i) $B(0) = 0$, (ii) $B(t)$ has stationary and independent increments, and (iii) $B(t)$ is normally distributed with mean value 0 and variance t (for $t > 0$). It can be shown that $B(t)$ has continuous realizations (with probability 1).

a) For $a > 0$, let $T_a = \inf\{t | B(t) \geq a\}$. That is, T_a denotes the first time $B(t)$ hits the level a. Use the relation

$$P\{B(t) \geq a\} = P\{B(t) \geq a | T_a \leq t\} P\{T_a \leq t\} + P\{B(t) \geq a | T_a > t\} P\{T_a > t\},$$

to show that $P\{T_a \leq t\} = 2P\{B(t) \geq a\}$. From this result, write down the cumulative distribution function for T_a in terms of the cumulative distribution function $\Phi(\cdot)$ of an $N(0,1)$ variable.

b) For $a > 0$, determine the cumulative distribution function of $\max_{0<s\leq t}\{B(s)\}$ in terms of $\Phi(\cdot)$.

c) Show that $P\{T_a < \infty\} = 1$, while $\mathrm{E}[T_a] = \infty$ $(a > 0)$.

Formulas for TMA4265 Stochastic Processes :

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$
$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^n$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{og} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \le t \le \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$\begin{aligned} P_{i,i+1}(h) &= \lambda_i h + o(h) \\ P_{i,i-1}(h) &= \mu_i h + o(h) \\ P_{i,i}(h) &= 1 - (\lambda_i + \mu_i)h + o(h) \\ P_{ij}(h) &= o(h) \quad \text{for } |j-i| \ge 2 \end{aligned}$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \ge 0$ are birth rates, $\mu_i \ge 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \ i \ne j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \ne j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \ne i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \ne j} q_{kj} P_k \quad \text{og} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{og} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{og} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[S W_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad ,$$

Differential equation

The differential equation $f'(t) + \alpha f(t) = g(t)$ for $t \geq 0$ with initial condition $f(0) = a$ has the solution

$$f(t) = a e^{-\alpha t} + \int_0^t e^{-\alpha(t-s)} g(s)\, ds$$