Exercises from the text book

3.30

- Let $X_0, X_1, \ldots$ be iid with $P(X_i = j) = p(j)$, where $\sum_{j=1}^{m} p(j) = 1$
- Let $N = \min\{N > 0 : X_n = X_0\}$

We have

$$\begin{aligned}
P(N > n) &= P\{X_1 \neq X_0, X_2 \neq X_0, \ldots, X_n \neq X_0\} \\
&= \sum_{j=1}^{m} P\{X_1 \neq X_0, X_2 \neq X_0, \ldots, X_n \neq X_0 \mid X_0 = j\} \cdot P(X_0 = j) \\
&= \sum_{j=1}^{m} (1 - p(j))^n \cdot p(j) \\
&\Downarrow \\
E[N] &= \sum_{n=0}^{\infty} P(N > n) = \sum_{n=0}^{\infty} \sum_{j=1}^{m} (1 - p(j))^n p(j) \\
&= \sum_{j=1}^{m} p(j) \sum_{n=0}^{\infty} (1 - p(j))^n \\
&= \sum_{j=1}^{m} p(j) \cdot \frac{1}{p(j)} = \underline{\underline{m}}
\end{aligned}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad x < 1$$

3.37

Let the number of errors X given that typist $T = A, B, C$ is typing be Poisson distributed with

$$X|T = A \sim \text{Poisson}(2.6)$$
$$X|T = B \sim \text{Poisson}(3.0)$$
$$X|T = C \sim \text{Poisson}(3.4)$$

From the Poisson distribution it is known that

$$E[X|T = A] = \text{Var}(X|T = A) = 2.6$$
$$E[X|T = B] = \text{Var}(X|T = B) = 3.0$$
$$E[X|T = C] = \text{Var}(X|T = C) = 3.4$$

Each of the typists is equaly likely to do the work, which means that

$$P(T = A) = P(T = B) = P(T = C) = \frac{1}{3}.$$

a)

$$\begin{aligned} E[X] &= \sum_t E[X|T = t]\, P(T = t) \\ &= E[X|T = A]\, P(T = A) + E[X|T = B]\, P(T = B) + E[X|T = C]\, P(T = C) \\ &= 2.6 \cdot \frac{1}{3} + 3.0 \cdot \frac{1}{3} + 3.4 \cdot \frac{1}{3} \\ &= \underline{\underline{3}} \end{aligned}$$

b) We use the relation

$$\text{Var}(X) = E[X^2] - E[X]^2$$

where

$$\begin{aligned} E[X^2] &= \sum_t E[X^2|T = t]\, P(T = t) \\ &= E[X^2|T = A]\, P(T = A) + E[X^2|T = B]\, P(T = B) + E[X^2|T = C]\, P(T = C) \\ &= (\text{Var}(X|T = A) + E[X|T = A]^2)\, P(T = A) \\ &+ (\text{Var}(X|T = B) + E[X|T = B]^2)\, P(T = B) \\ &+ (\text{Var}(X|T = C) + E[X|T = C]^2)\, P(T = C) \\ &= (2.6 + 2.6^2 + 3.0 + 3.0^2 + 3.4 + 3.4^2)\frac{1}{3} \\ &= 12.11 \end{aligned}$$

Thus we have

$$\begin{aligned} \text{Var}(X) &= E[X^2] - E[X]^2 \\ &= 12.11 - 3^2 \\ &= \underline{\underline{3.11}} \end{aligned}$$

3.38

U is uniform on $[0,1]$, hence $E(U) = \frac{1}{2}$ and $Var(U) = \frac{1}{12}$.

We perform n trials, where the probability of success is dependent on the value U takes. Let X be the number of successes on the n trials. Conditionled on $U = u$, X will be Binomial distributed with parameter $p = u$, that is, $X|U = u \sim \text{bin}(x; n, u)$. This gives $E(X|U) = nU$ and $Var(X|U) = nU(1-U)$.

We will find the expectation of the number of successes,

$$E(X) = E(E(X|U)) = E(nU) = nE(U) = \underline{\underline{\frac{n}{2}}}$$

and the variance of the number of successes

$$\begin{aligned} Var(X) &= E(Var(X|U)) + Var(E(X|U)) \\ &= E(nU(1-U)) + Var(nU) \\ &= \int_0^1 nu(1-u)du + n^2 Var(U) \\ &= n\int_0^1 udu - n\int_0^1 u^2 du + n^2 \frac{1}{12} \\ &= \frac{n}{2} - \frac{n}{3} + \frac{n^2}{12} = \underline{\underline{\frac{n}{6} + \frac{n^2}{12}}} \end{aligned} \tag{1}$$

3.41

Let N denote the number of minutes the rat is trapped in the maze. The rat chooses to go to the right or the left with equal probability. If it goes to the left, it either departs the maze or returns to the initial position. We define three choices $V = 1, 2, 3$, where

$$P(V = 1) = P(\text{Rat goes to the right}) = \frac{1}{2}$$
$$P(V = 2) = P(\text{Rat goes to the left, then departs}) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$$
$$P(V = 3) = P(\text{Rat goes to the left, then returns}) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}.$$

Then we have

$$\begin{aligned} E[N] &= \sum_{i=1}^{3} E[N|V = i] \cdot P(V = i) \\ &= (3 + E[N]) \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} + (5 + E[N]) \cdot \frac{1}{6} \end{aligned}$$

such that

$$E[N] = \underline{\underline{21}}$$

4.1

See solution in the book.

4.2

Whether or not it rains today depends on the weather conditions through the last three days. As rain today is only dependent on the past through the weather the last three days, the process follows a Markov chain model. Define

$$R : \text{Rain}$$
$$O : \text{Not rain}$$

We therefore have eight possible states

$$\Omega = \{RRR, RRO, ROR, ORR, ROO, ORO, OOR, OOO\}$$

4.3

The states are defined as in Exercise 4.2. We get the transition matrix

$P =$

	RRR	RRO	ROR	ORR	ROO	ORO	OOR	OOO
RRR	0.8	0.2	0.0	0.0	0.0	0.0	0.0	0.0
RRO	0.0	0.0	0.4	0.0	0.6	0.0	0.0	0.0
ROR	0.0	0.0	0.0	0.6	0.0	0.4	0.0	0.0
ORR	0.6	0.4	0.0	0.0	0.0	0.0	0.0	0.0
ROO	0.0	0.0	0.0	0.0	0.0	0.0	0.4	0.6
ORO	0.0	0.0	0.4	0.0	0.6	0.0	0.0	0.0
OOR	0.0	0.0	0.0	0.6	0.0	0.4	0.0	0.0
OOO	0.0	0.0	0.0	0.0	0.0	0.0	0.2	0.8