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Solution - Exercise 4

Exercises from the text book

4.18

Coin number one comes up heads with probability 0.6 and coin number two with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

Define state 0 to flip coin number one, and state 1 to flip coin number two. Transition matrix

$$P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$$

a) Want to know what proportion of flips we use coin number one.

We find the stationary probabilities by solving

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, j \geq 0, \quad \text{where} \quad \sum_{j=0}^{\infty} \pi_j = 1$$

In this case we get

$$\begin{aligned} \pi_0 &= 0.6\pi_1 + 0.5\pi_1 \\ \pi_1 &= 0.4\pi_0 + 0.5\pi_1 \\ \pi_0 + \pi_1 &= 1 \end{aligned}$$

By solving the system of equations, we get $\pi_0 = \frac{5}{9}$ and $\pi_1 = \frac{4}{9}$. This means that we will use coin number one in $\underline{\underline{\frac{5}{9}}}$ of the flips.

b) If we start the process with coin number one, what is the probability that coin number two is used in the fifth flip?

We have to find P_{01}^4.

$$P^2 = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix}$$

$$P^4 = P^2 P^2 \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix} \begin{bmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{bmatrix} = \begin{bmatrix} 0.5556 & 0.4444 \\ 0.5555 & 0.4445 \end{bmatrix}$$

We see that the probability is $\underline{\underline{P_{01}^4 == 0.4444 \approx \frac{4}{9}}}$.

4.20

- P is doubly stochastic $\Longleftrightarrow \sum_i P_{ij} = 1 \ \forall j$
- Let the chain be irreducible and aperiodic, and consist of $M+1$ states $0, 1, ..., M$
- Show that the limiting distributions are given by:

$$\pi_j = \frac{1}{M+1} \quad j = 0, ..., M \tag{1}$$

- As the chain is aperiodic and irreducible, the limiting distributions are equal to the stationary distributions which fulfil:
  i)
  $$\pi_j = \sum_i \pi_i P_{ij}$$
  ii)
  $$\sum_i \pi_i = 1$$

- We see that by putting $\pi_j = \frac{1}{M+1} \ \forall j$, we obtain:
  i)
  $$\pi_j = \sum_i \pi_i P_{ij} = \sum \frac{1}{M+1} P_{ij} = \frac{1}{M+1} \cdot \sum_i P_{ij} = \underline{\underline{\frac{1}{M+1}}}$$
  ii)
  $$\sum_i \pi_i = \sum_{i=0}^{M} \frac{1}{M+1} = \frac{1}{M+1} \cdot (M+1) = \underline{\underline{1}}$$

- Therefore (1) is the limiting distributions of the Markov chain.

4.25

- Let X_n be the number of pairs of shoes at the front door at time n. We have that:

$$\begin{aligned} P_{01} &= P(X_n = 1 \mid X_{n-1} = 0) \\ &= P(\text{Departs from back door, enters at front door} \mid X_{n-1} = 0) = \frac{1}{2} \cdot \frac{1}{2} = \underline{\frac{1}{4}} \\ &\Downarrow \\ P_{00} &= 1 - P_{01} = \underline{\frac{3}{4}} = P_{kk} \end{aligned}$$

$$0 < i < k \begin{cases} P_{i-1,i} = P(\text{Departs from back door, enters at front door} \mid X_{n-1} = i-1) = \frac{1}{2} \cdot \frac{1}{2} = \underline{\frac{1}{4}} \\ P_{i,i-1} = P(\text{Departs from front door, enters at back door} \mid X_{n-1} = i-1) = \frac{1}{2} \cdot \frac{1}{2} = \underline{\frac{1}{4}} \\ P_{ii} = 1 - (\frac{1}{4} + \frac{1}{4}) = \underline{\frac{1}{2}} \end{cases}$$

Transition matrix:

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & 0 \\ 0 & 0 & \dots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \dots & \dots & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

- The transition matrix is doubly stochastic (see exercise 4.20), hence $\pi_j = \frac{1}{k+1}$ $\forall j$ (P is both irreducible and aperiotic)
- We have:

$$\begin{aligned} P(\text{runs barefooted}) &= P(X_n = 0 \mid \text{front door}) \cdot P(\text{front door}) + P(X_n = k \mid \text{back door}) \cdot P(\text{back door}) \\ &= \pi_0 \cdot \frac{1}{2} + \pi_k \cdot \frac{1}{2} \\ &= \frac{1}{k+1} \cdot \frac{1}{2} + \frac{1}{k+1} \cdot \frac{1}{2} \\ &= \underline{\underline{\frac{1}{k+1}}} \end{aligned}$$

4.32

See solution in the book.

Exercises from exams

Eks. Jan. 98, oppg. 2

a)

When $X_n = 0$ we need to have $X_{n+1} \sim \text{bin}(12, 0.06)$ which means

$$\begin{aligned} P(X_{n+1} = 2 \mid X_n = 0) &= \binom{12}{2} \cdot 0.06^2 \cdot 0.94^{10} \\ &= \underline{\underline{0.128}} \end{aligned}$$

similarly

$$\begin{aligned} P(X_{n+1} = 0 \mid X_n = 2) &= \binom{10}{0} \cdot 0.06^0 \cdot 0.94^{10} \cdot \binom{2}{0} \cdot 0.5^0 \cdot 0.5^2 \\ &= \underline{\underline{0.135}} \end{aligned}$$

Let A_n be the number of members who both have a car on day n and want a car on day $n+1$.

$$\begin{aligned} P(X_{n+1} = 1 \mid X_n = 1) &= P(X_{n+1} = 1 \mid X_n = 1, A_n = 0) \cdot P(A_n = 0 \mid X_n = 1) \\ &\quad + P(X_{n+1} = 1 \mid X_n = 1, A_n = 1) \cdot P(A_n = 1 \mid X_n = 1) \\ &= \binom{11}{1} \cdot 0.06^1 \cdot 0.94^{10} \cdot 0.5 + \binom{11}{0} \cdot 0.06^0 \cdot 0.94^{11} \cdot 0.5 \\ &= \underline{\underline{0.431}} \end{aligned}$$

b)
W_n: amount payed by the members on day n.

$$W_n = \begin{cases} X_n \cdot 125 & if\ X_n \leq 2 \\ 2 \cdot 125 & if\ X_n = 3 \end{cases}$$

In the long run:

$$\begin{aligned} E[W_n] &= 0 \cdot \pi_0 + 125 \cdot \pi_1 + 250 \cdot \pi_2 + 250 \cdot \pi_3 \\ &= \underline{\underline{136.25}} \end{aligned}$$

Number of cars rented each day in the long run is:

$0 \cdot \pi_0 + 1 \cdot \pi_1 + 2 \cdot \pi_2 + 2 \cdot \pi_3 = 1.09$

Per wants to use his part of this, that is, the number of days Per uses a car is:

$$P(\text{Per uses}) = \frac{1.09}{12} = \underline{\underline{0.0908}}$$

$$P(\text{Per uses} \mid \text{Per wishes}) = \frac{P(\text{Per uses} \cap +\text{Per wishes})}{P(\text{Per wishes})} = \frac{P(\text{Per uses})}{P(\text{Per wishes})}$$

Have to find $P(\text{Per wishes})$.

$P(\text{Per wishes on day } n)$
$= P(\text{Per wishes on day } n \mid \text{Per uses on day } n-1) \cdot P(\text{Per uses on day } n-1)$
$+ P(\text{Per wishes on day } n \mid \text{Per does not use on day } n-1) \cdot P(\text{Per does not use on day } n-1)$
$= 0.5 \cdot \frac{1.09}{12} + 0.06 \cdot (1 - \frac{1.09}{12})$
$= \underline{\underline{0.100}}$

such that :

$$P(\text{Per uses} \mid \text{Per wishes}) = \frac{0.0908}{0.100} = \underline{\underline{0.909}}$$

c)
The behaviour of the members becomes dependent upon each other. Define:

$$U_n = \begin{cases} 1 & \text{if a certain member wishes to use a car} \\ 0 & \text{otherwise} \end{cases}$$

This is a Markov chain with transition matrix

$$P = \begin{bmatrix} 0.94 & 0.06 \\ 0.5 & 0.5 \end{bmatrix}$$

The limiting distribution is given by:

$$\left.\begin{array}{l} \pi_0 = 0.94 \cdot \pi_0 + 0.5 \cdot \pi_1 \\ \pi_1 = 0.06 \cdot \pi_0 + 0.5 \cdot \pi_1 \\ 1 = \pi_0 + \pi_1 \end{array}\right\} \Rightarrow \pi_0 = 0.8929 \quad \pi_1 = 0.1071$$

Let Y_n be the number of members who whish a car on day n. In the long run we get:
$Y_n \sim \text{bin}(12, 0.1071)$
such that:
$P(Y_n = 0)$=0.2568
$P(Y_n = 1)$=0.3697
$P(Y_n = 2)$=0.2439
$P(Y_n \geq 3)$=0.1297

Let:
V_n: net income at day n, that is,

$$V_n = \begin{cases} 125 \cdot Y_n & if\ Y_n \leq 2 \\ 125 \cdot Y_n - 250 \cdot (Y_n - 2) & if\ Y_n \geq 3 \end{cases}$$

$$= \begin{cases} 125 \cdot Y_n & if\ Y_n \leq 2 \\ 500 - 125 \cdot Y_n & if\ Y_n \geq 3 \end{cases}$$

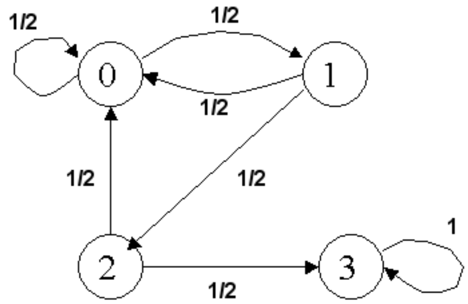
In the long run:

$$\begin{aligned} E[V_n] &= 125 \cdot P(Y_n = 1) + 250 \cdot P(Y_n = 2) \\ &\quad + \sum_{i=3}^{12} (500 - 250 \cdot i) \cdot P(Y_n = i) \\ &= 125 \cdot P(Y_n = 1) + 250 \cdot P(Y_n = 2) \\ &\quad + 500 \cdot P(Y_n \geq 3) - 125 \cdot \underbrace{\sum_{i=3}^{12} i \cdot P(Y_n = i)}_{=E[Y_n] - P(Y_n=1) - 2 \cdot P(Y_n=2)} \\ &= 125 \cdot 0.3697 + 250 \cdot 0.2439 + 500 \cdot 0.1297 - 125 \cdot (12 \cdot 0.1071 - 0.3697 - 2 \cdot 0.2439) \\ &= \underline{\underline{118.58}} \end{aligned}$$

Eks. Aug. 98, oppg. 1

a)

$$\begin{aligned} Pr\{X_2 = 1 | X_0 = 0\} &= Pr\{X_1 = 0 \cap X_2 = 1 | X_0 = 0\} \\ &= Pr\{X_1 = 0 | X_0 = 0\} \cdot Pr\{X_2 = 1 | X_1 = 0\} \\ &= \frac{1}{2} \cdot \frac{1}{2} = \underline{\underline{\frac{1}{4}}} \end{aligned}$$



$$\begin{aligned}
Pr\{X_3 = 0|X_0 = 0\} &= Pr\{X_1 = 0 \cap X_2 = 0 \cap X_3 = 0|X_0 = 0\} \\
&+ Pr\{X_1 = 0 \cap X_2 = 1 \cap X_3 = 0|X_0 = 0\} \\
&+ Pr\{X_1 = 1 \cap X_2 = 0 \cap X_3 = 0|X_0 = 0\} \\
&+ Pr\{X_1 = 1 \cap X_2 = 2 \cap X_3 = 0|X_0 = 0\} \\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \underline{\underline{\frac{1}{2}}}
\end{aligned}$$

$$\begin{aligned}
Pr\{X_4 = 3|X_0 = 0\} &= Pr\{X_1 = 0 \cap X_2 = 1 \cap X_3 = 2 \cap X_4 = 3|X_0 = 0\} \\
&+ Pr\{X_1 = 1 \cap X_2 = 2 \cap X_3 = 3 \cap X_4 = 3|X_0 = 0\} \\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot 1 = \underline{\underline{\frac{3}{16} = 0,1875}}
\end{aligned}$$

b) Let $v_i = E[T|X_0 = i], T = \min_{n\geq 0}\{n : X_n = 3\}$

First step analysis gives the equations

$$\begin{aligned} v_0 &= \frac{1}{2}v_0 + \frac{1}{2}v_1 + 1 \\ v_1 &= \frac{1}{2}v_0 + \frac{1}{2}v_2 + 1 \\ v_2 &= \frac{1}{2}v_0 + \frac{1}{2}v_3 + 1 \\ v_3 &= 0 \end{aligned}$$

which give

$$\begin{aligned} v_2 &= \frac{1}{2}v_0 + 1 \\ v_1 &= \frac{1}{2}v_0 + \frac{1}{2}(\frac{1}{2}v_0 + 1) + 1 = \frac{3}{4}v_0 + \frac{3}{2} \\ v_0 &= \frac{1}{2}v_0 + \frac{1}{2}(\frac{3}{4}v_0 + \frac{3}{2}) + 1 = \frac{7}{8}v_0 + \frac{7}{4} \end{aligned}$$

$\Rightarrow v_0 = \frac{7}{4} \cdot 8 = \underline{\underline{14}}$

c) Let Y_n be the number of the last flips who coincides with the start of the sequence MKK and get the transition matrix

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Firs step analysis gives the equations:

$$\begin{aligned} v_0 &= \frac{1}{2}v_0 + \frac{1}{2}v_1 + 1 \\ v_1 &= \frac{1}{2}v_1 + \frac{1}{2}v_2 + 1 \\ v_2 &= \frac{1}{2}v_1 + \frac{1}{2}v_3 + 1 \\ v_3 &= 0 \end{aligned}$$

which give

$$
\begin{aligned}
v_2 &= \frac{1}{2}v_1 + 1 \\
v_1 &= \frac{1}{2}v_1 + \frac{1}{2}(\frac{1}{2}v_1 + 1) + 1 = \frac{3}{4}v_1 + \frac{3}{2} \\
&\Rightarrow\ v_1 = \frac{3}{2} \cdot 4 = 6 \\
v_0 &= \frac{1}{2}v_0 + \frac{1}{2} \cdot 6 + 1 \\
&\Rightarrow\ v_0 = 4 \cdot 2 = \underline{\underline{8}}
\end{aligned}
$$

It is reasonable that this solution is less than in b) as you return to state 0 if you get a "wrong flip"in X_n, and as you only return to state 1 with a "wrong flip"in Y_n.

d) The system of equations for the stationay distribution

$$
\begin{aligned}
\pi_0 &= \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_3 \\
\pi_1 &= \frac{1}{2}\pi_0 \\
\pi_2 &= \frac{1}{2}\pi_1 \\
\pi_0 &+ \pi_1 + \pi_2 + \pi_3 = 1
\end{aligned}
$$

such that

$$\pi_0 = \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{8}\pi_0 + \frac{1}{2}\pi_3$$

$$\Rightarrow \frac{1}{8}\pi_0 = \frac{1}{2}\pi_3 \Rightarrow \pi_3 = \frac{1}{4}\pi_0$$

and

$$\pi_0 + \frac{1}{2}\pi_0 + \frac{1}{4}\pi_0 + \frac{1}{4}\pi_0 = 2\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{2}$$

$$\underline{\underline{\pi_0 = \frac{1}{2}, \pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{8}, \pi_3 = \frac{1}{8}}}$$

The chain has a limiting distribution because it is irreducible and aperiodic ($P_{00} > 0$). The limiting distribution is equal to the stationary distribution given above.

In the long run we will obtain MMM as the three final flips after $\frac{1}{8}$ of the flips.