Exercises from the text book

5.29

Kidney transplant

$T_A \sim \exp(\mu_A)$
$T_B \sim \exp(\mu_B)$

$T_1 =$ Time when new kidney arrives $\sim \exp(\lambda)$
$T_2 =$ Waiting time between arrival of first and second kidney $\sim \exp(\lambda)$ (as arrival of kidneys is Poisson process)

Rules:
i) First kidney goes to A (to B if A i dead)
ii) Second kidney goes to B (if B is still alive)

a)

$$Pr(\text{A gets new kidney}) = Pr(T_1 < T_A) = \underline{\underline{\frac{\lambda}{\lambda + \mu_A}}}$$

b)

$$\begin{aligned}
Pr(\text{B gets new kidney}) &= Pr(T_B > T_1)P(T_A < T_1) + P(T_B > T_1 + T_2)P(T_A > T_1) \\
&= Pr(T_B > T_1)P(T_A < T_1) + P(T_B > T_1 + T_2 | T_B > T_1)P(T_B > T_1)P(T_A > T_1) \\
&= Pr(T_B > T_1)P(T_A < T_1) + P(T_B > T_2)P(T_B > T_1)P(T_A > T_1) \\
&= \underline{\underline{\frac{\lambda}{\mu_B + \lambda} \cdot \frac{\mu_A}{\mu_A + \lambda} + \left(\frac{\lambda}{\mu_B + \lambda}\right)^2 \cdot \frac{\lambda}{\mu_A + \lambda}}}
\end{aligned}$$

5.42

$$\begin{aligned} N(t) &\sim \text{Poisson}(\lambda t) \\ S_n &= \text{Time of } n\text{'th event} \sim \text{Gamma}(\lambda,\ n) \\ E[S_n] &= \frac{n}{\lambda} \end{aligned}$$

a)

$$E[S_4] = \underline{\underline{\frac{4}{\lambda}}}$$

b)

$$\begin{aligned} E[\,S_4 \,|\, N(1) = 2\,] &= E[\,T_1 + T_2 + T_3 + T_4 \,|\, N(1) = 2\,] \\ &= E[1 + T_3 + T_4] \\ &= \underline{\underline{1 + \frac{2}{\lambda}}} \end{aligned}$$

We know that T_3 has not yet occured, and as $T_3 \sim \exp(\lambda)$, this is like restarting with T_3.

c) Independent increments give

$$\begin{aligned} E[\,N(4) - N(2) \,|\, N(1) = 3] &= E[\,N(4) - N(2)\,] \\ &= E[\,N(4-2)\,] = E[\,N(2)\,] \\ &= \underline{\underline{2\lambda}} \end{aligned}$$

5.60

Customers arrive at a bank at a Poisson rate λ. Suppose two customers arrived during the first hour, that is, $N(1) = 2$.

We need to know that $P(N(t+s) - N(s) = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$.

What is the probability that:

a) Both arrived during the first 20 minutes?

We need to find:

$$\begin{aligned} P(N(\frac{1}{3})=2 \mid N(1)=2) &= \frac{P(N(\frac{1}{3})=2 \cap N(1)=2)}{P(N(1)=2)} \\ &= \frac{P(N(\frac{1}{3})=2)\cdot P(N(\frac{2}{3})=0)}{P(N(1)=2)} \\ &= \frac{e^{-\lambda\frac{1}{3}}\frac{(\lambda\frac{1}{3})^2}{2!}e^{-\lambda\frac{2}{3}}\frac{(\lambda\frac{2}{3})^0}{0!}}{e^{-\lambda}\frac{\lambda^2}{2!}} \\ &= \frac{e^{-\lambda}\left(\frac{1}{3}\right)^2\lambda^2}{e^{-\lambda}\lambda^2} = \left(\frac{1}{3}\right)^2 = \underline{\underline{\frac{1}{9}}} \end{aligned}$$

b) At least one arrived during the first 20 minutes?

We have two possibilities. Either both arrived during the first 20 minutes, or only one arrived during the first 20 minutes and the second during the 40 last minutes. The first possibility was calculated in a). The second is:

$$\begin{aligned} P(N(\frac{1}{3})=1 \mid N(1)=2) &= \frac{P(N(\frac{1}{3})=1 \cap N(1)=2)}{P(N(1)=2)} \\ &= \frac{P(N(\frac{1}{3})=1)\cdot P(N(\frac{2}{3})=1)}{P(N(1)=2)} \\ &= \frac{e^{-\lambda\frac{1}{3}}\frac{(\lambda\frac{1}{3})^1}{1!}e^{-\lambda\frac{2}{3}}\frac{(\lambda\frac{2}{3})^1}{1!}}{e^{-\lambda}\frac{\lambda^2}{2!}} \\ &= \frac{\frac{2}{9}e^{-\lambda}\lambda^2}{\frac{1}{2}e^{-\lambda}\lambda^2} = \frac{4}{9} \end{aligned}$$

The probability that at least one arrived during the first 20 minutes is then $\frac{1}{9}+\frac{4}{9}=\underline{\underline{\frac{5}{9}}}$

5.62

N = number of errors in text $\sim$ Poisson(λ)
P_i = probability that proofreader i finds the error, i = 1,2

X_1 = number of errors found by proofreader 1, but <u>not</u> by 2
X_2 = number of errors found by proofreader 2, but <u>not</u> by 1
X_3 = number of errors found by both
X_4 = number of errors <u>not</u> found

a) $X_1,...,X_4$ have marginals like shown in b). Similarly as in exercise 5.44c), they are independent and have simultaneous distribution

$$P_r\{X_1=x_1,...,X_4=x_4\} = P_r\{X_1=x_1\}\cdot\ldots\cdot P_r\{X_4=x_4\}$$

b)

We have

$X_1 \mid N = n \sim Bin(n,p) \qquad p = p_1(1-p_2)$

$\Downarrow$

$$\begin{aligned}
&X_1 \sim Poisson(\lambda \cdot p) && \Rightarrow \quad E[X_1] = \lambda p_1(1-p_2) && \\
&X_2 \sim Poisson(\lambda \cdot p_2(1-p_1))) && \Rightarrow \quad E[X_2] = \lambda p_2(1-p_1) && \\
&\vdots && \quad\quad\; E[X_3] = \lambda p_1 p_2 && (1) \\
&X_4 \sim Poisson(\lambda \cdot (1-p_1)(1-p_2)) && \Rightarrow \quad E[X_4] = \lambda(1-p_1)(1-p_2) && (2)
\end{aligned}$$

This gives $\frac{E[X_1]}{E[X_3]} = \underline{\underline{\frac{1-p_2}{p_2}}}$ og $\frac{E[X_2]}{E[X_3]} = \underline{\underline{\frac{1-p_1}{p_1}}}$

c)

Estimator:

Put in $X_1, ..., X_3$ for $E[X_1], ..., E[X_3]$ in the answers from b), and get

$\underline{\underline{\hat{p_1} = \frac{X_3}{X_2+X_3}}}$ and $\underline{\underline{\hat{p_2} = \frac{X_3}{X_1+X_3}}}$

From (1) we obtain

$\underline{\underline{\hat{\lambda}}} = \frac{X_3}{\hat{p_1}\hat{p_2}} = ... = \underline{\underline{X_1 + X_2 + X_3 + \frac{X_1 \cdot X_2}{X_3}}}$

d)

(2) gives: $\underline{\underline{\hat{X_4}}} = \hat{\lambda}(1-\hat{p_1})(1-\hat{p_2}) = ... = \underline{\underline{\frac{X_1 \cdot X_2}{X_3}}}$

5.75

"Single-server station".

V_i = time between successive arrivals $\sim F(iid)$
S_i = service time $\sim G(iid)$
X_n = number of customers in system immediately before n'th arrival
Y_n = number of customers in system immediately after n'th departure

We have

$$X_n = X_{n-1} + 1 - D_n \quad \text{and} \quad \mathrm{Y_n} = \begin{cases} Y_{n-1} - 1 + A_n & Y_{n-1} \geq 1 \\ A_n & Y_{n-1} = 0 \end{cases} \qquad (\star)$$

where

D_n = number of customers being serviced during n'th arrival time V_n
A_n = number of customers arriving during n'th departure time S_n

Demand:
$\{X_n\}$ Markov chain $\Leftrightarrow X_n \mid X_{n-1}$ independent of $\{X_k\}_{k \leq n-2}$

a) If $V_i \sim F = exp(\lambda)$, that is, forgetful, then A_n is independent of all events before Y_{n-1}, that is, independent of $\{Y_k\}_{k \leq n-2}$. (A_n is, however, dependent of S_n).

From ($\star$) we know that $Y_n \mid Y_{n-1}$ is independent of $\{Y_k\}_{k \leq n-2} \Rightarrow \{Y_n\}$ is a Markov chain.

(If F was not forgetful, A_n would have been dependent of how much time before Y_{n-1} the last arrival appeared; denote this time by T. If $(Y_{n-k}, ..., Y_{n-1}) = (r+k, ..., r)$, that is, if no one arrives during the last k departure times, there is a large probability that T is large. Therefore A_n is dependent of $\{Y_k\}_{k \leq n-2} \Rightarrow \{Y_n\}$ NOT Markov!!!

$\{Y_k\}_{k \leq 2} \overset{independent}{\longleftrightarrow} T \overset{independent}{\longleftrightarrow} A_n$)

b) If $S_i \sim G = exp(\mu)$, that is, forgetful, D_n is independent of everything that happened prior to X_{n-1}, that is, independent of $\{X_k\}_{k \leq n-2}$

From ($\star$) we know that $X_n \mid X_{n-1}$ is independent of $\{X_k\}_{k \leq n-2} \Rightarrow \{X_n\}$ is a Markov chain.

Using a simular argument as in a), we obtain that $\{X_n\}$ is *not* a Markov chain if G is *not* forgetful.

c) From a) we know that when $F \sim exp(\lambda)$, then $\{Y_n\}$ is a Markov chain, and ($\star$) gives that

$$\begin{aligned} P_r\{Y_n = k \mid Y_{n-1} = j\} &= P_r\{Y_{n-1} - 1 + A_n = k \mid Y_{n-1} = j\} \\ &= P_r\{A_n = k - j + 1 \mid Y_{n-1} = j\} = P_r\{A_n = k - j + 1\} \end{aligned}$$

(A_n is independent of Y_{n-1}, $j \geq 1$)

The low of total probability gives that

$$P_r\{A_n = k'\} = \int_0^\infty P_r\{A_n = k' \mid S_n = s\} \cdot g_{s_n}(s)ds = \int_0^\infty \frac{(\lambda s)^{k'}}{k'!} e^{-\lambda s} \cdot g_{s_n}(s)ds$$

The transition probabilities for $\{Y_n\}$ are

$$P_{Y_n}(j,k) = \begin{cases} \int_0^\infty \frac{(\lambda s)^{k-j+1}}{(k-j+1)!} e^{-\lambda s} g(s)ds & j \geq 1, k \geq j-1 \\ \int_0^\infty \frac{(\lambda s)^k}{k!} e^{-\lambda s} g(s)ds & j = 0, k \geq j \end{cases}$$

Similarly we have that when $G\sim exp(\mu)$, $\{X_n\}$ is a Markov chain, and ($\star$) gives

$$\begin{aligned} P_r\{X_n = k \mid X_{n-1} = j\} &= P_r\{X_{n-1} + 1 - D_n = k \mid X_{n-1} = j\} \\ &= P_r\{D_n = j - k + 1 \mid X_{n-1} = j\} \end{aligned}$$

If we condition on the arrival time V_n, we get that

$$P_r\{D_n = k' \mid V_n = v, X_{n-1} = j\} = \left\{ \begin{array}{ll} \frac{(\mu v)^{k'}}{k'!} e^{-\mu v} & k' \leq j \\ \sum_{i=k'}^{\infty} \frac{(\mu v)^i}{i!} e^{-\mu v} & k' = j+1 \\ 0 & ellers \end{array} \right\} = P_{j,k'}(v)$$

Therefore
$P_{X_n}(j,k) = \int_0^\infty P_{j,j-k+1}(v) \cdot f_{v_n}(v) dv$

6.1

- Given that we have n males and m females,

$$Pr\{\text{male } i \text{ meets female } j \text{ turing time period } h\ \} = \lambda h + o(h)$$

The number of offspring these two produce is then a Poisson process with rate λ, and the waiting time before the first birth is

$$T_{ij} \sim exp(\lambda)$$

The waiting time before the first birth in the entire population is
$T = \min_{i,j} \sim exp(n \cdot m \cdot \lambda)$
(There are $n \cdot m$ combinations of males and females.)

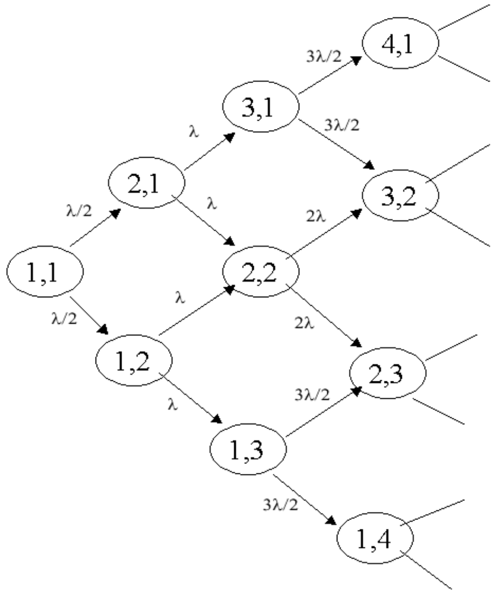
- [i)]

  In state $\{n, m\}$ we have

$$\begin{aligned} v_{\{n,m\}}^{-1} &= E[\text{Time in state } \{n,m\} \text{ before moving on}] \\ &= E[T] \\ &= \frac{1}{nm\lambda} \quad \Rightarrow \quad v_{\{n,m\}} = nm\lambda \end{aligned}$$

2. The probability of giving birth to a male and a female is equal, hence

$$P_{\{n,m\},\ \{n+1,m\}} = P_{\{n,m\},\ \{n,m+1\}} = \frac{1}{2}$$



Alternatively:

$$
\begin{aligned}
Pr\{\text{Male } i \text{ does NOT meet female } j \text{ during } h\} &= 1 - \lambda h + o(h) \\
&\Downarrow \\
Pr\{\text{No male meets a female during } h\} &= (1 - \lambda h + o(h))^{nm} \\
&= 1 - nm\lambda h + o(h) \\
&= \underline{P_{ii}(h)}
\end{aligned}
$$

We have that

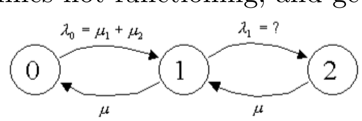
$$\nu_i = \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \lim_{h \to 0} \frac{nm\lambda h + o(h)}{h} = \underline{\underline{nm\lambda}}$$

6.2

See solution in the book.

6.3

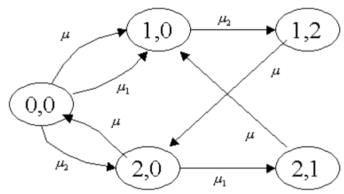
- Analyze by the number of machines not functioning, and get



  BUT we can not analyze this any further as λ_1 is dependent of *which* machine breakes down first.

- We can alternatively consider the states

$$X_t = \begin{cases} \text{(0,0) - Both machines functioning} \\ \text{(1,0) - Machine 1 not functioning, M2 functioning.} \\ \text{(2,0) - Machine 2 not functioning, M1 functioning} \\ \text{(1,2) - M1 being repaird, M2 not functioning.} \\ \text{(2,1) - M2 being repaird, M1 not functioning.} \end{cases}$$



- We get

$$P = \begin{array}{c} \\ (0,0) \\ (1,0) \\ (2,0) \\ (1,2) \\ (2,1) \end{array} \begin{array}{c} \begin{array}{ccccc} (0,0) & (1,0) & (2,0) & (1,2) & (2,1) \end{array} \\ \begin{pmatrix} 0 & \frac{\mu_1}{\mu_1+\mu_2} & \frac{\mu_2}{\mu_1+\mu_2} & 0 & 0 \\ \frac{\mu}{\mu+\mu_2} & 0 & 0 & \frac{\mu_2}{\mu+\mu_2} & 0 \\ \frac{\mu}{\mu+\mu_1} & 0 & 0 & 0 & \frac{\mu_1}{\mu+\mu_1} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad \underline{v} = \begin{pmatrix} \mu_1+\mu_2 \\ \mu_2+\mu \\ \mu_1+\mu \\ \mu \\ \mu \end{pmatrix}$$

6.4

See solution in the book.

Exercises from exams

Eksamen, mai '03 oppg.2

a)

$$\begin{aligned} P\{N(h)=1\} &= P\{N_A(h)+N_B(h)=1\} = P\{(N_A(h)=1 \cap N_B(h)=0) \cup (N_A(h)=0 \cap N_B(h)=1)\} \\ &= P\{N_A(h)=1 \cap N_B(h)=0\} + P\{N_A(h)=0 \cap N_B(h)=1\} \text{ (disjoint events)} \\ &= P\{N_A(h)=1\}P\{N_B(h)=0\} + P\{N_A(h)=0\}P\{N_B(h)=1\} \text{ (as } N_A(h) \text{ and } N_B(h) \text{ are independent)} \\ &= (\lambda_A h + o(h))(1-\lambda_B h + o(h)) + (1-\lambda_A h + o(h))(\lambda_B h + o(h)) \\ &= \lambda_A h - \lambda_A\lambda_B h^2 + o(h) + \lambda_B h - \lambda_A \lambda_B h^2 + o(h) \\ &= \underline{\underline{(\lambda_A+\lambda_B)h + o(h)}} \text{ (as } h^2 \text{ is a } o(h)\text{-function).} \end{aligned}$$

Showing that $\mathrm{P}\{N(h) \geq 2\} = o(h)$ is easiest by first considering $\mathrm{P}\{N(h) = 0\}$:

$$\mathrm{P}\{N(0) = 0\} = \mathrm{P}\{N_A(h) = 0 \cap N_B(h) = 0\} = \mathrm{P}\{N_A(h) = 0\}\mathrm{P}\{N_B(h) = 0\}$$

$$= (1 - \lambda_A h + o(h))(1 - \lambda_B h + o(h))$$

$$= 1 - \lambda_B h - \lambda_A h + \lambda_A \lambda_B h^2 + o(h)$$

$$= 1 - (\lambda_A + \lambda_B)h + o(h) \text{ (siden } h^2 \text{ er en } o(h)\text{-funksjon).}$$

This gives

$$\mathrm{P}\{N(h) \geq 2\} = 1 - \mathrm{P}\{N(h) = 0\} - \mathrm{P}\{N(h) = 1\}$$

$$= 1 - (1 - (\lambda_A + \lambda_B)h + o(h)) - ((\lambda_A + \lambda_B)h + o(h)) = \underline{\underline{o(h)}}.$$

b)

Use that $N(t)$ is a Poisson process with intensity $\lambda = \lambda_A + \lambda_B = 3$:

$$\mathrm{P}\{N(1) \leq 2\} = \mathrm{P}\{N(1) = 0\} + \mathrm{P}\{N(1) = 1\} + \mathrm{P}\{N(1) = 2\}$$

$$= \frac{\lambda^0}{0!}e^{-\lambda} + \frac{\lambda^1}{1!}e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} = e^{-\lambda}(1 + \lambda + \frac{1}{2}\lambda^2) = \underline{\underline{0.423}}.$$

Use definition of conditional probability to get

$$\mathrm{P}\{N_A(1) = 1 | N(1) = 1\} = \frac{\mathrm{P}\{N_A(1) = 1, N(1) = 1\}}{\mathrm{P}\{N(1) = 1\}} = \frac{\mathrm{P}\{N_A(1) = 1, N_B(1) = 0\}}{\mathrm{P}\{N(1) = 1\}}$$

$$= \frac{\mathrm{P}\{N_A(1) = 1\}\mathrm{P}\{N_B(1) = 0\}}{\mathrm{P}\{N(1) = 1\}} \text{ (as } N_A(1) \text{ and } N_B(1) \text{ are independent)}$$

$$= \frac{\frac{\lambda_A^1}{1!}e^{-\lambda_A} \cdot \frac{\lambda_B^0}{0!}e^{-\lambda_B}}{\frac{\lambda^1}{1!}e^{-}\lambda} = \frac{\lambda_A}{\lambda_A + \lambda_B} = \underline{\underline{\frac{1}{3}}} \text{ (as } \lambda = \lambda_A + \lambda_B).$$

Use that $N(t) = N_A(t) + N_B(t)$ to get

$$\mathrm{P}\{N(1) \leq 2 | N_A(0.5) = 1\} = \mathrm{P}\{N_B(0.5) + (N(1) - N(0.5)) \leq 1 | N_A(0.5) = 1\}$$

$$= \mathrm{P}\{N_B(0.5) + (N(1) - N(0.5)) \leq 1\} \text{ (as } N_A(0.5) \text{ is independent of } N_B(0.5) \text{ and } N(1) - N(0.5))$$

$$= \mathrm{P}\{N_B(0.5) = 0, N(1) - N(0.5) = 0\} + \mathrm{P}\{N_B(0.5) = 1, N(1) - N(0.5) = 0\}$$

$$+ \mathrm{P}\{N_B(0.5) = 0, N(1) - N(0.5) = 1\}$$

$$= e^{-\lambda_B \cdot 0.5} \cdot e^{-\lambda \cdot 0.5} + \lambda_B \cdot 0.5 \cdot e^{-\lambda_B \cdot 0.5} \cdot e^{-\lambda \cdot 0.5} + e^{-\lambda_B \cdot 0.5} \cdot \lambda \cdot 0.5 \cdot e^{-\lambda \cdot 0.5}$$

$$= e^{-(\lambda + \lambda_B) \cdot 0.5}(1 + 0.5 \cdot \lambda_B + 0.5 \cdot \lambda) = \underline{\underline{0.2873}}.$$