Problem 1

Let X_n, $n = 0, 1, 2, \ldots$ be a Markov chain with state space $S = \{0, 1, 2, 3, 4\}$ and transition probability matrix

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix} \end{array}$$

a)
- Which properties must be fulfilled, so that you know that a Markov chain has a limiting distribution?
- Are these properties fulfilled here?

b) Compute the limiting distribution.

Problem 1

Assume the transition probability matrix is changed to

$$\tilde{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix} \end{array}$$

c)
- Justify why this Markov chain has no limiting distribution.

  The Markov chain can still have a stationary distribution. If so, compute it. What is the interpretation of this distribution?

Problem 2: Chess

The two chess players Magnus and Vishy are playing a series of games against each other.

http://thecityschool.edu.pk/liaquat-campus-chess-champions-sindh-festival/

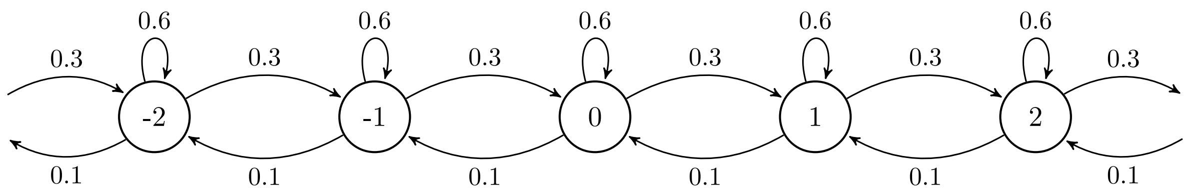
Setting

We assume that the probabilities for the outcomes of one game are given by the following:

- $P(\text{Magnus wins}) = 0.3$
- $P(\text{Vishy wins}) = 0.1$
- $P(\text{The game is a draw}) = 0.6$

Let $\{X_n\}_{n=0}^{\infty}$ be a stochastic process, where X_n denotes the number of victories of Magnus minus the number of victories for Vishy, after they have played n games.

Potential questions of interest



Let $X_0 = 0$

a) Explain why the stochastic process $\{X_n\}_{n=0}^{\infty}$ is a Markov chain. Also explain whether the chain is irreducible. Is the chain aperiodic?

b) Calculate the probability that the first three games are draws?

Calculate the probability that the two players will be even after three games?

Potential questions of interest (II)

c) Apply a first step analysis and find the probability that the chain will reach state 2 before it reaches state -2.

(This means that Magnus is the first player to be in the lead with two more victories than his opponent.)

Potential questions of interest (III)

We now assume that the states -3 and 3 are absorbing and define P_T for the states $\{-2, -1, 0, 1, 2\}$.

$$\mathbf{S} = (\mathbf{I} - \mathbf{P}_T)^{-1} = \begin{Vmatrix} 3.32 & 3.30 & 3.21 & 2.97 & 2.23 \\ 1.10 & 4.40 & 4.29 & 3.96 & 2.97 \\ 0.36 & 1.43 & 4.64 & 4.29 & 3.21 \\ 0.11 & 0.44 & 1.43 & 4.40 & 3.30 \\ 0.03 & 0.11 & 0.36 & 1.10 & 3.32 \end{Vmatrix} \quad \text{where} \quad \mathbf{P}_T = \begin{Vmatrix} 0.6 & 0.3 & 0 & 0 & 0 \\ 0.1 & 0.6 & 0.3 & 0 & 0 \\ 0 & 0.1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.1 & 0.6 & 0.3 \\ 0 & 0 & 0 & 0.1 & 0.6 \end{Vmatrix}$$

d) What is the expected number of games played before one of the players is in the lead with three victories?

What is the probability that Vishy is never in the lead?