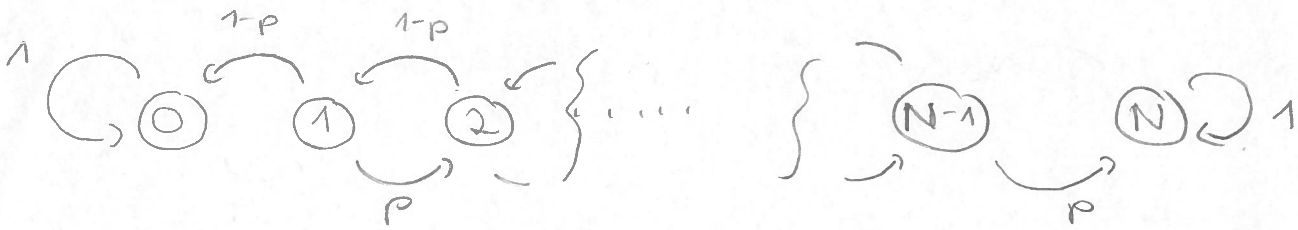
4.5.1. Gambler's ruin



$0 < p < 1 \qquad p = P(\text{Win one game})$

X_n = Gain after the n-th game

Equivalence classes $\{0\}$, $\{1, \dots, N-1\}$, $\{N\}$

recurrent $d=1$ transient $d=2$ recurrent $d=1$

Since each transient state will be visited finitely often the gambler will either attain N or go broke.

What is the probability that starting with i units the gambler's fortune will reach N before reaching 0.

Let $T = \min\{n \geq 0, X_n = 0 \text{ or } X_n = N\}$

$p_i = P(X_T = N \mid X_0 = i)$

Obviously $p_0 = 0 \quad p_N = 1$

$$p_i = P(X_T = N \mid X_0 = i)$$

$$= \sum_{j=0}^{N} P(X_T = N \mid X_1 = j, X_0 = i)\, P(X_1 = j \mid X_0 = i)$$

$$= \sum_{j=0}^{N} P(X_T = N \mid X_0 = j)\, \underbrace{P_{ij}}_{0 \text{ except for } j=i-1,\ j=i+1}$$

$$p_i = p_{i-1}\,(1-p) + p_{i+1} \cdot p$$

$$p_i\,(1-p) + p_i p = p_i = p_{i-1}(1-p) + p_{i+1} p$$

$$\Downarrow$$

$$\underbrace{(p_i - p_{i-1})}_{x_i}\,(1-p) = p\,\underbrace{(p_{i+1} - p_i)}_{x_{i+1}}$$

$$x_{i+1} = \frac{1-p}{p}\,x_i = r x_i \qquad i = 1, \dots, N-1$$

Let us look at $x_2 = r x_1 = r p_1$ $\quad (p_0 = 0)$

$$x_3 = r x_2 = r^2 p_1$$

$$\vdots$$

$$x_i = r^{i-1} p_1 \qquad i = 1, \dots, N$$

Note: $\displaystyle\sum_{k=1}^{i} x_k = (p_1 - p_0) + (p_2 - p_1) + \dots + (p_i - p_{i-1})$

$$= p_i$$

$$\Rightarrow \quad p_i = \sum_{k=1}^{i} x_k = \sum_{k=1}^{i} r^{k-1}\, p_1$$

We know (geometric series)

$$\sum_{k=0}^{n-1} a^k = \begin{cases} \frac{1-a^n}{1-a} & \text{if } a \neq 1 \\ n & \text{if } a = 1 \end{cases}$$

$\Rightarrow$ for $i = 1, 2, \dots, N$

$$p_i = \begin{cases} \frac{1-r^i}{1-r} p_1 & p \neq \frac{1}{2} \Leftrightarrow r \neq 1 \\ i p_1 & p = \frac{1}{2} \Leftrightarrow r = 1 \end{cases}$$

Using the fact that $p_N = 1$ we obtain

$$p_N = 1 = \begin{cases} \frac{1-r^N}{1-r} p_1 & p \neq \frac{1}{2} \\ N p_1 & p = \frac{1}{2} \end{cases}$$

so that (solve by p_1)

$$p_1 = \begin{cases} \frac{1-r}{1-r^N} & p \neq \frac{1}{2} \\ \frac{1}{N} & p = \frac{1}{2} \end{cases}$$

and

$$\underline{\underline{p_i}} = \begin{cases} \frac{1-r^i}{1-r^N} & p \neq \frac{1}{2} \\ \frac{i}{N} & p = \frac{1}{2} \end{cases}$$

For $N \to \infty$:

$$\lim_{N \to \infty} p_i = \begin{cases} 1 - r^i & r < 1 \Leftrightarrow p > \frac{1}{2} \\ 0 & r > 1 \Leftrightarrow p < \frac{1}{2} \\ 0 & p = \frac{1}{2} \end{cases}$$

If $p \leq \frac{1}{2}$ the gambler will, with probability 1, go broke against an infinitely rich adversary.

If $p > \frac{1}{2}$ there is a positive probability that the gambler's fortune will increase indefinitely.

$p = 0.501$
$i = 100$ (start amount) $\Rightarrow 1 - r^i \approx 0.33$

$i = 1000 \Rightarrow 1 - r^i \approx 0.98$

prob. being infinitely rich!

$\Rightarrow$ look at example 4.28 and clinical trials afterwards.