NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Processes**

Academic contact during examination: Andrea Riebler
Phone: 4568 9592

Examination date: December 6th 2014
Examination time (from–to): 09:00–13:00
Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottman: Matematisk formelsamling.
- Dictionary in any language.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers should be justified.
In your solution you can use English and/or Norwegian.

Language: English
Number of pages: 7
Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1

Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.4 \\ 0.25 & 0.25 & 0.5 & 0 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{pmatrix} \end{array}$$

a)

- Assume that the initial state distribution for the Markov chain is

  $$P(X_0 = 0) = \frac{1}{2}, \qquad P(X_0 = 1) = P(X_0 = 2) = P(X_0 = 3) = \frac{1}{6}$$

  Compute the unconditional probability $P(X_1 = 3)$.
- This Markov chain has a limiting distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \pi_3)$. How can π_i be interpreted? Mention two ways.

b) Derive the probability that state 3 is entered before state 2, if $X_0 = 0$.

Problem 2

Aliens from two planets (planet Zeeba and planet X) are arriving on Earth independently according to Poisson processes $\{N_Z(t); t \geq 0\}$ and $\{N_X(t); t \geq 0\}$ with parameters λ_Z and λ_X, respectively. The Extraterrestrials Arrival Registration Service Authority (EARSA) will begin registering alien arrivals at $t = 0$. Assume that $N_Z(0) = N_X(0) = 0$.

Let T_1 denote the time EARSA will function until it registers its first alien. Let Z be the event that the first alien to be registered is from planet Zeeba. Let T_2 be the time EARSA will function until at least one alien from both planets is registered.

For all following parts, you need to **show and explain your derivations!**

a)
- What is the probability distribution of $N(t) = N_Z(t) + N_X(t)$? Express $P(N(1) \leq 2)$ in terms of λ_Z and λ_X.
- Express $\mu_1 = \mathrm{E}(T_1)$ in terms of λ_Z and λ_X.

b)
- Express $p = P(Z)$ in terms of λ_Z and λ_X.
- Assume $\lambda_Z = 1$ per month and $\lambda_X = 2$ per month. Determine the probability that at least two of the first five arriving aliens are from planet Zeeba.

c) Express $\mu_2 = \mathrm{E}(T_2)$ in terms of λ_Z and λ_X.

d) Explain how you would check your result in part c) using simulations on the computer. Be explicit enough so that someone else (who knows nothing about stochastic processes) could implement it according to your instructions. You can also use pseudo-code.

Problem 3

The PhD students of the statistics group at the Department of Mathematics at NTNU would like to take part in a beach volleyball tournament. Unfortunately, each member of the statistics team gets sporadically sick. Assume that the complete team consists of n players. Further, assume that the time until a fit team member gets sick is exponentially distributed with parameter μ (independently of the state of the other players). On the other hand, the time until a sick team member gets fit again to play volleyball is exponentially distributed with parameter λ (independently of the state of the other players).

a) Model the situation described as a birth-death process, where the states denote the number of players that are fit.
Sketch the transition graph and provide all birth and death rates.

b) Show that the probability that exactly i players are fit (in the long-run), is given by

$$P_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n}, \qquad i = 0, \ldots, n,$$

where $\rho = \lambda/\mu$.

(Hint: $\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n$.)

c)
- Do the limiting probabilities P_i, $i = 0, \ldots, n$, also exist, if $\lambda \geq \mu$? Please explain.
- Assume that the statistics team has $n = 5$ players, and that $\lambda^{-1} = 4$ and $\mu^{-1} = 2$. Calculate the probability that the statistics team cannot participate at the tournament. (For participation at least two fit players are required.)

Problem 4

Suppose that you own one share of a stock whose price changes according to a Brownian motion process $\{X(t), t \geq 0\}$ with variance parameter $\sigma^2 = 4$. Assume time is measured in months.

a)
- Compute
$$P(X(13) \geq 11 \mid X(9) = 8)$$
- Suppose that you purchased the share at a price of 14, and the present time price is 11. You decide to sell the share either when it reaches the price 14 or when 4 months go by (whichever occurs first). What is the probability that you do not recover your purchase price?

Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1, S_2, \ldots, S_n | N(t)}(s_1, s_2, \ldots, s_n | n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$

$$P_{i,i-1}(h) = \mu_i h + o(h)$$

$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$

$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \;\; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$