TMA4265-Solution sketch - December exam HS2014

December 18, 2014

Problem 1

a)
- $P(X_1 = 3) = \sum_{i=0}^{3} P(X_1 = 3 \mid X_0 = i)P(X_0 = i) = \frac{1}{2}\frac{1}{10} + \frac{1}{6}\frac{4}{10} + \frac{1}{6}\frac{3}{10} = \frac{1}{6}$.
- The primary interpretation of π is

$$\pi_j = \lim_{n\to\infty} P_{ij}^n$$

That means, after the process has been in operation for a long duration, the probability of finding the Markov chain in state j is π_j, irrespective of the initial state.

A second interpretation is that π_i represents the long-run mean fraction of time that the process is in state i.

b) Let $u_i = P(A \mid X_0 = i)$, where A represent the event that state 3 is entered before state 2. We have

$$\begin{aligned}
u_0 &= 0.4u_0 + 0.3u_1 + 0.2u_2 + 0.1u_3 \\
u_1 &= 0.2u_0 + 0.2u_1 + 0.2u_2 + 0.4u_3 \\
u_2 &= 0 \\
u_3 &= 1
\end{aligned}$$

so that

$$\begin{aligned}
u_0 &= 0.4u_0 + 0.3u_1 + 0.1 \\
u_1 &= 0.2u_0 + 0.2u_1 + 0.4
\end{aligned}$$

This leads to $u_1 = \frac{1}{4}u_0 + \frac{1}{2}$. Setting this in the equation for u_0, we get $u_0 = \frac{10}{21}$.

Problem 2

a) – $N(t)$ is a merged Poisson process of two independent Poisson processes. Thus the rate of the merged process is given by the sum of the individual rates, i.e the rate is $\lambda = \lambda_Z + \lambda_X$.

$$\begin{aligned} P(N(1) \leq 2) &= P(N(1) = 0) + P(N(1) = 1) + P(N(1) = 2) \\ &= \frac{\lambda^0}{0!}\exp(-\lambda) + \frac{\lambda^1}{1!}\exp(-\lambda) + \frac{\lambda^2}{2!}\exp(-\lambda) \\ &= \exp(-\lambda)(1 + \lambda + \frac{1}{2}\lambda^2) \end{aligned}$$

– We consider again the process $N(t)$ of arrivals of aliens from both planets. Since this is a merged Poisson process with arrival rate $\lambda = \lambda_Z + \lambda_X$, the time until the first arrival is therefore exponentially distributed with parameter λ. Thus, $\mu_1 = E(T_1) = \frac{1}{\lambda} = \frac{1}{\lambda_Z + \lambda_X}$.
Alternatively, consider $T_1 = \min(T_1^Z, T_1^X)$ where T_1^Z and T_1^X are the first arrival times of aliens from planet Zeeba and planet X, respectively. Thus, T_1^Z and T_1^X are exponentially distributed with parameters λ_Z and λ_X, respectively. The minimum is therefore as well exponentially distributed with parameter $\lambda_Z + \lambda_X$, so that $\mu_1 = E(T_1) = \frac{1}{\lambda} = \frac{1}{\lambda_Z + \lambda_X}$.

b) – We consider the same merged process as before, with arrival rate $\lambda_Z + \lambda_X$. An arrival is with probability $\frac{\lambda_Z}{\lambda_Z + \lambda_X}$ from planet Zeeba and with probability $\frac{\lambda_X}{\lambda_Z + \lambda_X}$ from planet X. The question asked for $P(Z) = \frac{\lambda_Z}{\lambda_Z + \lambda_X}$. Short proof:

$$\begin{aligned} P(Z) &= P(N_Z(t) = 1, N_X(t) = 0 \mid N(t) = 1) \\ &= \frac{P(N_Z(t) = 1) \cdot P(N_X(t) = 0)}{P(N(t) = 1)} \\ &= \frac{(\lambda_Z \cdot t)^1 \exp(-\lambda_Z t) \cdot (\lambda_X \cdot t)^0 \exp(-\lambda_X t)}{((\lambda_X + \lambda_Z) \cdot t)^1 \exp(-(\lambda_Z + \lambda_X) \cdot t)} \\ &= \frac{\lambda_Z}{\lambda_X + \lambda_Z} \end{aligned}$$

– The number of aliens from planet Zeeba among the first 5 arriving aliens follows a binomial distribution with parameters (5, P(Z)). The reason is that all arrivals are independent and the probability that an arriving alien is from planet Zeeba is constant. Thus, letting $\lambda_Z = 1$ and $\lambda_X = 2$ we get $P(Z) = \frac{1}{3}$ and it follows:

$$1 - \left(\frac{2}{3}\right)^5 - 5 \cdot \left(\frac{2}{3}\right)^4 \frac{1}{3} = 0.539$$

c) The time T_2 until at least one alien from planet Zeeba and one alien from planet X has arrived can be expressed as $T_2 = \max(T_1^Z, T_1^X)$ where T_1^Z and T_1^X are the first arrival times of aliens from planet Zeeba and planet

X, respectively. That mean, T_1^Z and T_1^X are exponentially distributed with parameters λ_Z and λ_X, respectively.

The expected time until the first alien arrives was calculated in a), $\mu_1 = E(T_1) = \frac{1}{\lambda} = \frac{1}{\lambda_Z+\lambda_X}$. To compute the remaining time we condition on the 1st alien being from planet Zeeba (e.g. event Z) or planet X (event Z^C), and use

$$\begin{aligned}
E(T_2) = E(T_1) + P(Z)E(\text{time until first X-alien arrive} \mid Z) + \\
P(Z^C)E(\text{time until first Zeeba-alien arrive} \mid Z^C) \\
= E(T_1) + P(Z)E(T_1^X) + (1 - P(Z))E(T_1^Z) \\
= \frac{1}{\lambda_Z + \lambda_X} + \frac{\lambda_Z}{\lambda_Z + \lambda_X}\left(\frac{1}{\lambda_X}\right) + \frac{\lambda_X}{\lambda_Z + \lambda_X}\left(\frac{1}{\lambda_Z}\right)
\end{aligned}$$

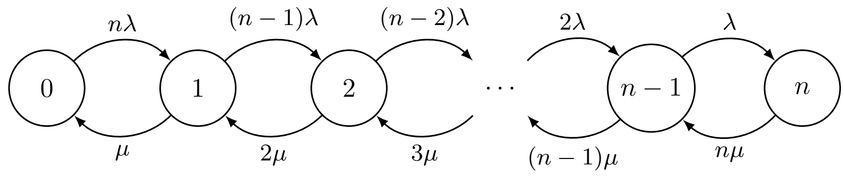
d) One potential algorithm:

- We need several simulations in order to get the expected value.
- In each simulation:
  * we simulate the interarrival time from an exponential distribution with rate $\lambda_Z + \lambda_X$.
  * we decide whether an arrival comes from Zeeba or planet X by sampling from a Bernoulli distribution with probability $P(Z)$.
  * we stop as soon as we have seen both arrivals and return the arrival time (sum of the interarrival times) of the last event.
- We compute the average over all returned arrival times.

Note: This solution is not unique!

Problem 3

a) The transition graph is as follows



The death rates are

$$\begin{aligned} \mu_0 &= 0, \\ \mu_i &= i\mu \qquad \text{for } i = 1, \dots, n \end{aligned}$$

The birth rates are

$$\begin{aligned} \lambda_i &= (n-i)\lambda, \qquad \text{for } i = 0, \dots, n-1 \\ \lambda_n &= 0 \end{aligned}$$

Here we used that the minimum of i independent and exponentially distributed (with parameter λ) random variables is an exponentially distributed random variable with parameter $i\lambda$.

b) Let P_i denote the state of i in the long run, which is given by

$$P_i = \theta_i P_0$$

where

$$\theta_i = \frac{(n\lambda) \cdot (n-1)\lambda \cdots (n-i+1)\lambda}{\mu \cdot 2\mu \cdots i\mu} = \binom{n}{i} \left(\frac{\lambda}{\mu}\right)^i$$

Using that $\frac{\lambda}{\mu} = \rho$, we have $\theta_i = \binom{n}{i}\rho^i$. We further know that

$$\begin{aligned} P_0 &= \frac{1}{\sum_{i=0}^{n} \theta_i} \\ &= \frac{1}{\sum_{i=0}^{n} \binom{n}{i}\rho^i} \\ &= \frac{1}{(1+\rho)^n} \end{aligned}$$

Thus, it follows that

$$P_i = \frac{\binom{n}{i}\rho^i}{(1+\rho)^n}.$$

c)
- In general, if $\lambda \geq \mu$, an M/M/1 queue might grow infinitely and therefore does not reach a stationary distribution. This cannot happen in this birth-and-death process, because the number of states is bounded.

– We know that $n = 5$, $\lambda^{-1} = 4$ and $\mu^{-1} = 2$, so that $\rho = 1/2$. We calculate the probability that there are less than two fit players as

$$\begin{aligned}
P_0 + P_1 &= \frac{1}{(1+\rho)^5}\,(1 + 5 \cdot \rho) \\
&= \left(\frac{2}{3}\right)^5 \cdot \frac{7}{2} \\
&= \frac{32}{243} \cdot \frac{7}{2} \\
&= \frac{224}{486} \approx 0.46
\end{aligned}$$

Problem 4

We have that $\{B(t) : t \geq 0\}$ is a standard Brownian motion, where $B(t) = \frac{X(t)}{2}$. Thus

a) – We have

$$\begin{aligned}
P(X(13) \geq 11 \mid X(9) = 8) &= P(B(13) \geq \frac{11}{2} \mid B(9) = \frac{8}{2}) \\
&= P(B(13) - B(9) \geq 5.5 - 4 \mid B(9) = 4) \\
&\overset{\text{indep. increments}}{=} P(B(13) - B(9) \geq 1.5) \\
&\overset{\text{stat. increments}}{=} P(B(4) - B(0) \geq 1.5) \\
&\overset{B(0)=0}{=} P(B(4) \geq 1.5) = 1 - \Phi\left(\frac{1.5}{\sqrt{4}}\right) \\
&= 0.2266
\end{aligned}$$

– Let $T_{1.5}$ denote the time to hit 1.5. We are interested in

$$\begin{aligned}
P(\max_{0 \leq s \leq 4} X(s) < 3) &= 1 - P(\max_{0 \leq s \leq 4} X(s) \geq 3) \\
&= 1 - P(\max_{0 \leq s \leq 4} B(s) \geq 3/2) \\
&= 1 - P(T_{1.5} \leq 4) \\
&= 1 - [2\left(1 - \Phi\left(\frac{1.5}{\sqrt{4}}\right)\right)] = 1 - 0.4532 = 0.5468
\end{aligned}$$

The probability that you do not recover the purchase price is ≈ 0.55.