Problem 1.

a) • All lines need to sum to 1

$\Rightarrow x = 0.6, \ y = 0.1, \ z = 0.55$

• $P(X_3 = 0 | X_2 = 0, X_1 \neq 0, X_0 = 0) = P(X_3 = 0 | X_2 = 0) =$

$P(X_1 = 0 | X_0 = 0) = 0.4$

• $p = \frac{y}{y + 0.3} = \underline{\underline{\frac{1}{4}}}$

b) Chain is irreducible, aperiodic and pos. recurrent

$\Rightarrow P^T \pi = \pi \qquad \sum_{i=0}^{2} \pi_i = 1$ has a unique

solution

$\longrightarrow$

b)

$$\begin{pmatrix} 0.4 & 0.3 & 0 \\ 0.6 & 0.6 & 0.45 \\ 0 & 0.1 & 0.55 \end{pmatrix} \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix} = \begin{pmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{pmatrix}$$

$$\sum_{i=0}^{2} \pi_i = 1$$

$$0.4\,\pi_0 + 0.3\pi_1 = \pi_0 \quad \Rightarrow \quad 0.6\pi_0 = 0.3\pi_1$$

$$\Rightarrow \quad \pi_0 = \frac{1}{2}\pi_1$$

$$0.6\,\pi_0 + 0.6\,\pi_1 + 0.45\pi_2 = \pi_1$$

$$0.1\,\pi_1 + 0.55\,\pi_2 = \pi_2 \quad \Rightarrow \quad 0.45\pi_2 = 0.1\,\pi_1$$

$$\Rightarrow \pi_2 = \frac{2}{9}\pi_1$$

$$\Rightarrow \frac{1}{2}\pi_1 + \pi_1 + \frac{2}{9}\pi_1 \overset{!}{=} 1$$

$$\left(\frac{9+18+4}{18}\right)\pi_1 \overset{!}{=} 1$$

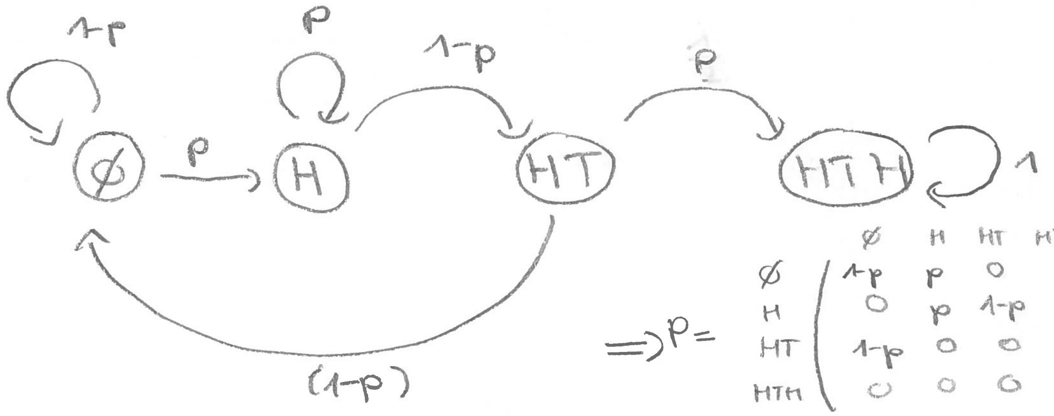
$$\frac{31}{18}\pi_1 = 1 \quad \Rightarrow \quad \pi_1 = \frac{18}{31}$$

$$\Rightarrow \quad \pi_0 = \frac{9}{31}$$

$$\Rightarrow \quad \pi_2 = \frac{4}{31}$$

$\Rightarrow$ The long run proportion of time Gary is in a glum mode is $\frac{4}{31} \sim 0.13 \approx 13\%$.

Problem 2



$$\Rightarrow P = \begin{array}{c|c} & \begin{array}{cccc} \emptyset & H & HT & HTH \end{array} \\ \begin{array}{c} \emptyset \\ H \\ HT \\ HTH \end{array} & \begin{pmatrix} 1-p & p & 0 & 0 \\ 0 & p & 1-p & 0 \\ 1-p & 0 & 0 & p \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$\Rightarrow$ ① $V_\emptyset = 1 + (1-p)V_\emptyset + pV_H$

② $V_H = 1 + pV_H + (1-p)V_{HT}$

③ $V_{HT} = 1 + (1-p)V_\emptyset + pV_{HTH}$

④ $V_{HTH} = 0$

$\overset{③}{\Rightarrow}$ $V_{HT} = 1 + (1-p)V_\emptyset$

$\overset{②}{\Rightarrow}$ $V_H = 1 + pV_{HH} + (1-p)[1+(1-p)V_\emptyset]$

$$V_H = \frac{1 + (1-p)[1+(1-p)V_\emptyset]}{(1-p)}$$

$\overset{①}{\Rightarrow}$ $V_\emptyset = 1 + (1-p)V_\emptyset + p\left(\frac{1}{1-p} + 1 + (1-p)V_\emptyset\right)$

$$V_\emptyset - (1-p)V_\emptyset - p(1-p)V_\emptyset = 1 + \frac{p}{1-p} + p$$

$$V_\emptyset(1 - 1 + p - p + p^2) = \frac{1 - \cancel{p} + \cancel{p} + p - p^2}{1-p}$$

$$V_\emptyset = \frac{1 + p - p^2}{p^2(1-p)}$$

For $p = \frac{1}{2}$ we get $V_\emptyset = 10$

Problem 3.)

In order to have a time-reversible MC we need to have

$$\pi_0 P_{01} = \pi_1 P_{10}$$

$$\pi_1 P_{12} = \pi_2 P_{21}$$

$$\pi_2 P_{20} = \pi_0 P_{02}$$

$\Rightarrow$ From this it directly follows that

$$P_{01} \cdot P_{12} \cdot P_{20} = P_{10} \cdot P_{02} \cdot P_{21}$$

needs to be fulfilled

Let $P_{01} = 0.7$, $P_{12} = 0.5$, $P_{20} = 0.6$

$P_{10} = 0.2$, $P_{02} = 0.2$, $P_{21} = 0.2$

we get $P_{01} \cdot P_{12} \cdot P_{20} = 0.21 \neq$

$P_{10} \cdot P_{02} \cdot P_{21} = \frac{1}{125} = 0.008$

The propability transition matrix is

$$P = \begin{pmatrix} 0.1 & 0.7 & 0.2 \\ 0.2 & 0.3 & 0.5 \\ 0.6 & 0.2 & 0.2 \end{pmatrix}$$

$\Rightarrow$ all states communicate

$\Rightarrow$ irreducible

finite state space

$\Rightarrow$ pos. recurrent

ergodic

$\Rightarrow$ one state with $P_{ii} > 0$

$\Rightarrow$ aperiodic

Problem 4

a) • $P(N(0.5) \geq 2) = 1 - P(N(0.5) \leq 1) = 1 - \left(\frac{(2 \cdot 0.5)^0 e^{-1}}{0!} + \frac{(2 \cdot 0.5)^1}{1!} e^{-1} \right)$

$= 1 - (e^{-1} + e^{-1}) \approx \underline{0.264}$

• $E(S_4 \mid N(1) = 3) = 1 + \frac{6}{2} = \underline{4}$

b) Catching of salmon is Poisson process (PP) with rate $\lambda_S = 0.4 \cdot 2 = 0.8$, equally catching trout is PP with $\lambda_T = 0.6 \cdot 2 = 1.2$.

$\Rightarrow$ The number of salmons $N_S(2.5)$ and the number of trouts $N_T(2.5)$ that David catches in the first 2.5 hours are independently Poisson distributed with rate $\underbrace{0.8 \cdot 2.5}_{2}$ and $\underbrace{1.2 \cdot 2.5}_{3}$ respectively.

$\Rightarrow$ P(1 salmon and 2 trouts) =

$P(N_S(2.5) = 1, N_T(2.5) = 2) = P(N_S(2.5) = 1) \cdot P(N_T(2.5) = 2)$

$\Rightarrow \frac{2^1 e^{-2}}{1!} \cdot \frac{3^2 e^{-3}}{2!} = 2e^{-2} \cdot 4.5 e^{-3} = \underline{9e^{-5}}$

$\approx \underline{0.0606}$

c) Here, we use the property that the time to catch the first fish is exponentially distributed. We also know that the minimum of n iid exponentially distributed random variables is exponentially distributed with rate $n \cdot \lambda$.

$\Rightarrow$ ~~Time~~ Expected time until first catch $= \frac{1}{3 \cdot \lambda} = \frac{1}{6}$

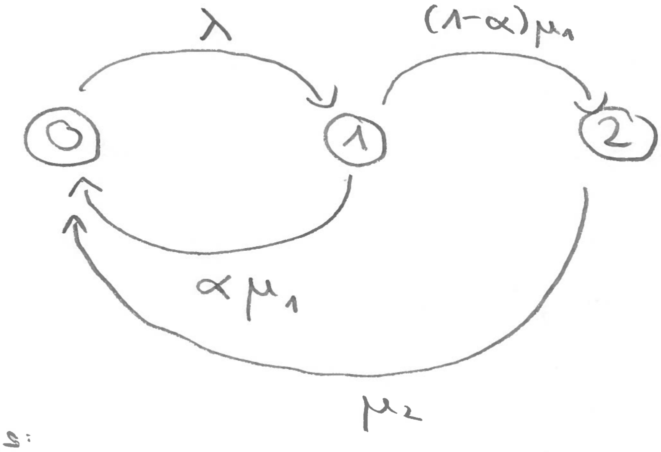
$\Rightarrow$ Two left to catch first fish $= \frac{1}{2 \cdot \lambda} = \frac{1}{4}$

$\Rightarrow$ Last one $= \frac{1}{2}$ $\Rightarrow E(T) = \frac{1}{6} + \frac{1}{4} + \frac{1}{2} = \underline{55 \text{ minutes}}$

Problem 5

a) Assume we have 3 states 0 (guide is free), 1 (guide is negotiating), 2 (guide is on tour).

$\Rightarrow$



Balance equations:

	leaving rate	=	arrival rate
0	λP_0	=	$\alpha \mu_1 P_1 + \mu_2 P_2$
1	$\mu_1 P_1$	=	λP_0
2	$\mu_2 P_2$	=	$(1-\alpha)\mu_1 P_1$

$$\Rightarrow \quad P_1 = \frac{\lambda}{\mu_1} P_0 \qquad P_2 = \frac{(1-\alpha)\mu_1}{\mu_2} \cdot P_1$$

$$= \frac{(1-\alpha)\cdot\lambda}{\mu_2} P_0$$

$$\Rightarrow \quad P_0 + \frac{\lambda}{\mu_1} P_0 + \frac{(1-\alpha)\cdot\lambda}{\mu_2} \cdot P_0 = 1$$

$$\Rightarrow P_0 \left(\frac{\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1(1-\alpha)}{\mu_1\mu_2} \right) = 1$$

$$\Rightarrow \quad P_0 = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1(1-\alpha)}$$

b) Embedded discrete time MC:

$$P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & \alpha & 0 & 1-\alpha \\ 2 & 1 & 0 & 0 \end{array}$$

where we assume $0 < \alpha < 1$

$\Rightarrow$ all states communicate $\Rightarrow$ one equivalence class. We need to check the period only for one state as it is a class property

$\Rightarrow$ $P_{00}^{1} = 0$ $\qquad \Rightarrow$ Period is equal to $\underline{\underline{1}}$

$P_{00}^{2} > 0$

$P_{00}^{3} > 0$

$P_{00}^{4} > 0$

$P_{00}^{5} > 0$

...

Problem 6

We start at $t=0$ with $X(0)=0$

Simulate $Z_0, \dots, Z_k \overset{iid}{\sim} N(0,1)$

$$X(t_0) = \sigma \cdot \sqrt{t_0} \cdot Z_0 + \mu t_0.$$

$$X(t_1) = X(t_0) + \sigma \cdot \sqrt{t_1 - t_0} \cdot Z_1 + \mu(t_1 - t_0)$$

$$\vdots$$

$$X(t_k) = \sum_{i=1}^{k} \left(\sigma \sqrt{t_i - t_{i-1}} \cdot Z_{i-1} + \mu(t_i - t_{i-1}) \right) \qquad k>0$$

Here, we use the property of stationary and independent increments.

where we know that $X(t) - X(s) \sim N(\mu(t-s), \sigma^2(t-s))$

for $t > s \geq 0$.