TMA4265 Stochastic Processes
Week 35 – Solutions

Problem 1.12: Probability of being first event to occur

The experiment consists of repeated trials where in each trial E, F or neither occurs, i.e. the outcome is selected from $\{E, F, (E \cup F)^{\mathcal{C}}\}$. In each repetition the probability of each of these events is the same.

For the experiment to consist of n repetitions and ending with E it is necessary that $(E \cup F)^{\mathcal{C}}$ occurs $n-1$ times and that E occurs the last time. Let N denote the random variable describing the total number of repetitions. We may then write

$$\mathrm{P}(E \cap (N = n)) = \mathrm{P}(E)(1 - \mathrm{P}(E) - \mathrm{P}(F))^{n-1}.$$

The desired probability is then found by summing over all possible values of n,

$$\begin{aligned}
\mathrm{P}(E \text{ occurs before } F) &= \sum_{n=1}^{\infty} \mathrm{P}(E)(1 - \mathrm{P}(E) - \mathrm{P}(F))^{n-1} \\
&= \frac{\mathrm{P}(E)}{\mathrm{P}(E) + \mathrm{P}(F)} \sum_{n=1}^{\infty} (1 - \mathrm{P}(E) - \mathrm{P}(F))^{n-1}(\mathrm{P}(E) + \mathrm{P}(F)) \\
&= \frac{\mathrm{P}(E)}{\mathrm{P}(E) + \mathrm{P}(F)},
\end{aligned}$$

where the last equality is done by recognizing the sum of all probabilities in a geometric distribution.

Problem 1.44: Draws from an urn

Let U be a random variable denoting which urn is chosen, with value 1 if Urn 1 is chosen and with value 2 if Urn 2 is chosen. Since the urn is determined by a fair coin we have $P(U = 1) = P(U = 2) = 0.5$. The question in the book can be rephrased as what is the probability that Urn 2 was selected given that a white ball was drawn. By Bayes's formula we find

$$\begin{aligned}
\mathrm{P}(U = 2 | \text{White ball drawn}) &= \frac{\mathrm{P}(U = 2 \cap \text{White ball drawn})}{\mathrm{P}(\text{White ball drawn})} \\
&= \frac{\mathrm{P}(\text{White ball drawn}|U = 2)\mathrm{P}(U = 2)}{\mathrm{P}(\text{White ball drawn}|U = 1)\mathrm{P}(U = 1) + \mathrm{P}(\text{White ball drawn}|U = 2)\mathrm{P}(U = 2)} \\
&= \frac{\frac{3}{3+12} \cdot \frac{1}{2}}{\frac{5}{5+7} \cdot \frac{1}{2} + \frac{3}{3+12} \cdot \frac{1}{2}} \\
&= \frac{24}{74}.
\end{aligned}$$

Problem 1.13: The dice game craps

Let F be a random variable denoting the sum achieved on the first throw. There are three situations based on the value of the first throw.

Case 1. If $F = 7$ or $F = 11$ the player wins immediately.

Case 2. If $F = 2$, $F = 3$ or $F = 12$ the player loses immediately.

Case 3. If F does not take one of the aforementioned values, then she continues throwing until she either throws the sum F again, and wins, or she throws a seven, and loses. This is exactly the situation in Problem 1.12. Thus we must calculate the probability of throwing a specific sum. It can be shown that

$$\mathrm{P}(F = i) = \begin{cases} \frac{i-1}{36}, & i = 2, \ldots, 7, \\ \frac{13-i}{36}, & i = 8, \ldots, 12, \end{cases}$$

by counting the number of ways to achieve each sum.

We find

$$\begin{aligned}
\text{P(Win)} &= P(F=7) + P(F=11) + \sum_{i\in\{4,5,6,8,9,10\}} P(F=i)\frac{P(F=i)}{P(F=i)+P(F=7)} \\
&= \frac{6}{36} + \frac{2}{36} + \frac{\left(\frac{3}{36}\right)^2}{\frac{3}{36}+\frac{6}{36}} + \frac{\left(\frac{4}{36}\right)^2}{\frac{4}{36}+\frac{6}{36}} + \frac{\left(\frac{5}{36}\right)^2}{\frac{5}{36}+\frac{6}{36}} + \frac{\left(\frac{5}{36}\right)^2}{\frac{5}{36}+\frac{6}{36}} + \frac{\left(\frac{4}{36}\right)^2}{\frac{4}{36}+\frac{6}{36}} + \frac{\left(\frac{3}{36}\right)^2}{\frac{3}{36}+\frac{6}{36}} \\
&= 0.493
\end{aligned}$$

Problem 2.43: Expected number of draws

1. X denotes the total number of red balls removed before the first black ball is chosen and X_i is an indicator variable for whether red ball i is removed before the first black ball is chosen. Therefore, X can be expressed as
$$X = X_1 + X_2 + \ldots + X_n.$$

2. We are first interested in $\text{E}[X_i] = P(X_i = 1)$. First, make the observation that the other red balls are irrelevant in calculating this probability. It does not matter whether they are drawn before or after red ball i. Second, ignoring the other red balls it is equally likely to draw red ball i or one of the black balls. We get
$$\text{E}[X_i] = P(X_i = 1) = \frac{1}{m+1}.$$
Since the expectation of a sum is the sum of the expectations, we find
$$\text{E}[X] = \text{E}[X_1] + \ldots + \text{E}[X_n] = \frac{n}{m+1}.$$