TMA4265 Stochastic Processes
Week 38 – Solutions

Problem 4.10

This problem can be solved by making "glum mood" an absorbing state and then calculating the three-step transition probabilities. The modified transition probability matrix is

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.0 & 0.0 & 1.0 \end{bmatrix}$$

and the three-step transition probability matrix is

$$\mathbf{P}^3 = \begin{bmatrix} 0.293 & 0.292 & 0.415 \\ 0.219 & 0.220 & 0.561 \\ 0 & 0 & 1.000 \end{bmatrix}.$$

The desired probability is then the probability of making a three-step transition from cheerful mood to either cheerful or so-so mood without being absorbed in the glum state, $0.293 + 0.292 = 0.585$.

4.16

See solution in the book

Exercise 1

Given a homogeneous Markov chain, where the transition matrix $\mathbf{P}$ depends on a parameter p given by

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.2 & p & 0 & 0.8-p \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0.1 & 0.1 & 0.8 \\ 0.1 & p & 0.1 & 0.8-p \end{pmatrix} \end{array}$$

For which value of p is the Markov chain not irreducible?

Solution

A Markov chain is called irreducible if all states are mutually accessible (communicate), i.e. the probability to get in finite number of steps from state i to state j is positive for all states $i, j \in S$. In the example given here this property is not fulfilled for $p = 0.8$, as state 3 and 4 cannot be reached from states 1 and 2.

Exercise 2

Consider a Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.6 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Let $\{X_n; n \geq 0\}$ be a Markov chain with the given transition probability and let $T = \min\{n \geq 0 | X_n = 0 \text{ or } X_n = 3\}$ denote the time of absorption:

a) We are interested in $P(X_T = 0|X_0 = 1)$ and use a first-step analysis to find this quantity. Let $u_i = P(X_T = 0|X_0 = i)$. Obviously $u_0 = 1$ and $u_3 = 0$. Further

$$\begin{aligned} u_1 &= \sum_{j=0}^{3} u_j P_{1j} \\ &= u_0 P_{10} + u_1 P_{11} + u_2 P_{12} \\ &= 0.1 + 0.4 u_1 + 0.1 u_2 \end{aligned}$$

$$\begin{aligned} u_2 &= \sum_{j=0}^{3} u_j P_{2j} \\ &= u_0 P_{20} + u_1 P_{21} + u_2 P_{22} \\ &= 0.2 + 0.1 u_1 + 0.6 u_2 \end{aligned}$$

Solving this linear system of two equations in two variables gives $u_1 = P(X_T = 0|X_0 = 1) \approx 0.261$.

b) We are interested in $E(T|X_0 = 1)$ and use a first-step analysis to find the desired quantity. Let $v_i = E(T|X_0 = i)$ Obviously $v_0 = v_3 = 0$. Further

$$\begin{aligned} v_1 &= 1 + \sum_{j=0}^{3} v_j P_{1j} \\ &= 1 + v_1 P_{11} + v_2 P_{12} \\ &= 1 + 0.4 v_1 + 0.1 v_2 \end{aligned}$$

$$\begin{aligned} v_2 &= 1 + \sum_{j=0}^{3} v_j P_{2j} \\ &= 1 + v_1 P_{21} + v_2 P_{22} \\ &= 1 + 0.1 v_1 + 0.6 v_2 \end{aligned}$$

Solving this system gives the mean time of absorption starting in state 1, $v_1 = E(T|X_0 = 1) \approx 2.17$.