Exercise 1

Show that for a Poisson process $N = \{N(t), t \geq 0\}$ the following statement is valid:

$$P(N(s) = k \mid N(t) = n) = \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}, \quad \text{for } s < t$$

Solution

Assume $s < t$ and assume the rate is λ, then by Bayes' law

$$P(N(s) = k \mid N(t) = n) = \frac{P(N(s) = k, N(t) = n)}{P(N(t) = n)}$$

The condition specified in the nominator of the fraction can be stated as $N(s) = k$ and $N(t) - N(s) = n - k$ and we find

$$P(N(s) = k \mid N(t) = n) = \frac{P(N(s) = k, N(t) - N(s) = n - k)}{P(N(t) = n)}$$

We use the independence of $N(s)$ and $N(t) - N(s)$ to find

$$\begin{aligned} P(N(s) = k \mid N(t) = n) &= \frac{P(N(s) = k) \cdot P(N(t) - N(s) = n - k)}{P(N(t) = n)} \\ &= \frac{e^{-s\lambda} \cdot \frac{(\lambda s)^k}{k!} \cdot e^{-(t-s)\lambda} \cdot \frac{(\lambda(t-s))^{n-k}}{(n-k)!}}{e^{-t\lambda} \cdot \frac{(\lambda t)^n}{n!}} \\ &= e^{-s\lambda-(t-s)\lambda+t\lambda} \cdot \lambda^{n-k+k-n} \cdot \frac{n!}{(n-k)!k!} \frac{s^k}{t^k} \cdot \frac{(t-s)^k}{t^k} \\ &= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k} \end{aligned}$$