TMA4265 Stochastic Processes
Week 35 – Solutions

Problem 3: Joint distribution

1.

$$
\begin{aligned}
p_X(x) &= \sum_y p(x,y) = \sum_y \exp(-2\lambda)\frac{\lambda^{x+y}}{x!\,y!} \\
&= \exp(-\lambda)\frac{\lambda^x}{x!}\underbrace{\sum_y \exp(-\lambda)\frac{\lambda^y}{y!}}_{1} \\
&= \exp(-\lambda)\frac{\lambda^x}{x!}
\end{aligned}
$$

Hence, X is Poisson distributed with parameter λ, e.g. $X \sim \mathcal{P}(\lambda)$. Analogously, $Y \sim \mathcal{P}(\lambda)$. We find that X and Y are independent, since $p(x,y) = p_X(x)p_Y(y)$ is fulfilled. Hence,

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y) \overset{X \text{ and } Y \text{ indep.}}{=} E(X)E(Y) - E(X)E(Y) = 0,$$

$\Rightarrow$ the covariance of X and Y is zero.

2. This is analogous to the proof in the lecture where we used that $X + Y \sim P(2\lambda)$. Hence

$$
\begin{aligned}
P(X|Z = X+Y) &= \frac{P(X = x, Z = x+Y)}{P(Z = z)} \\
&= \frac{P(X = x, Z = x+Y)}{P(Z = z)} \\
&= \frac{P(X = x, Y = z-x)}{P(Z = z)} \\
&= \frac{P(X = x)P(Y = z-x)}{P(Z = z)} \\
&= \frac{\exp(-\lambda)\frac{\lambda^x}{x!}\exp(-\lambda)\frac{\lambda^{z-x}}{(z-x)!}}{\exp(-2\lambda)\frac{(2\lambda)^z}{z!}} \\
&= \frac{\frac{\lambda^z}{x!(x-z)!}}{\frac{(2\lambda)^z}{z!}} \\
&= \frac{z!}{x!(x-z)!}\left(\frac{1}{2}\right)^z \\
&= \binom{z}{x}\left(\frac{1}{2}\right)^z \\
&= \binom{z}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{z-x}
\end{aligned}
$$

Hence, $X|X+Y$ is binomially distributed with size z and probability 0.5, e.g. $X|X+Y \sim \mathcal{B}(z, 0.5)$.

3.

$$\begin{aligned}\operatorname{Cov}(X+Y, X-Y) &= E((X+Y)(X-Y)) - E(X+Y)E(X-Y) \\ &= E(X^2 - Y^2) - (E(X)+E(Y)) \cdot \underbrace{(E(X)-E(Y))}_{0} \\ &= E(X^2) - E(Y^2) \\ &= 0\end{aligned}$$

Problem 4: Expectation

1. This exercise is a special case of Example 3.15 in the book. Let N_2 be the number of necessary rolls until two consecutive sixes appear, and let M_2 denote its mean. We condition on N_1 the number of trials needed for one six. Hence

$$M_2 = E(N_2) = E(E(N_2|N_1)),$$

where

$$\begin{aligned}E(N_2|N_1) &= \underbrace{p \cdot (N_1+1)}_{\text{case 1}} + \underbrace{(1-p) \cdot (N_1 + 1 + E(N_2))}_{\text{case 2}} \\ &= p \cdot N_1 + p + N_1 + 1 - p \cdot N_1 - p + (1-p)E(N_2) \\ &= N_1 + 1 + (1-p)E(N_2)\end{aligned}$$

It takes N_1 rolls to get one six, then either the next roll is a six (with probability $p = \frac{1}{6}$)) as well, and we are done (case 1), or it is not a six (with probability $1-p = \frac{5}{6}$) and we must begin anew (case 2). For case 2, it is important to have in mind that we have already needed $N_1 + 1$ rolls to get that far.

Taking expectations of both sides of the preceding yields

$$M_2 = M_1 + 1 + (1-p)M_2$$

or

$$M_2 = \frac{M_1 + 1}{p}.$$

Since N_1, the time of the first six, is geometric with parameter p we see that

$$M_1 = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6,$$

and thus

$$M_2 = \frac{6+1}{\frac{1}{6}} = 42.$$

The expected of rolls we need until the firs pair of consecutive sixes appears is 42.

Problem 2: Changes in stock prices

The model in the problem can be written as

$$Z = X_0 + \sum_{i=1}^{N} X_i,$$

where X_0 is always included and one or more X_i, for $i \geq 1$, may be included. The desired variance can be calculated via the law of total variance,

$$\begin{aligned}
\mathrm{Var}[Z] &= \mathrm{E}[\mathrm{Var}[Z|N]] + \mathrm{Var}[\mathrm{E}[Z|N]] \\
&= \mathrm{E}[\sigma^2 + N\sigma^2] + \mathrm{Var}[0 + N \cdot 0] \\
&= \sigma^2 + \nu\sigma^2 = (1+\nu)\sigma^2.
\end{aligned}$$