Problem 1

a)

$$P(X_2 = 2, X_1 = 1|X_0 = 1) = P(X_2 = 2|X_1 = 1, X_0 = 1)P(X_1 = 1|X_0 = 1)$$
$$= P(X_2 = 2|X_1 = 1)P(X_1 = 1|X_0 = 1) = 0.6 \cdot 0.4 = 0.24$$

$$P(X_2 = 2|X_0 = 1) = \sum_{k=1^2} P(X_2 = 2|X_1 = k)P(X_1 = k|X_0 = 1) = 0.6 \cdot 0.4 + 0.4 \cdot 0.6 = 0.48$$

The time until collapse is geometric distributed with 'success parameter $P(4, 1) = 0.6$. The expectation is $1/P(4, 1) = 1/0.6 = 1.67$.

b) Long-run distribution $\pi_j = \lim_{t\to\infty} P(X_t = j|X_0 = i)$.
We have $\pi_j = \sum_{i=1}^{4} \pi_i P(i, j)$.

$$\begin{aligned}
\pi_4 &= \pi_3 0.2 + \pi_4 0.4 \\
\pi_3 &= \pi_2 0.5 + \pi_3 0.4 \\
\pi_2 &= \pi_1 0.6 + \pi_2 0.4 \\
\pi_1 &= \pi_1 0.4 + \pi_2 0.1 + \pi_3 0.4 + \pi_4 0.6
\end{aligned}$$

$$\begin{aligned}
3\pi_4 &= \pi_3 \\
\frac{0.6}{0.5}\pi_3 &= \pi_2 \\
\pi_2 &= \pi_1 \\
\pi_1 &= \pi_1 0.4 + \pi_2 0.1 + \pi_3 0.4 + \pi_4 0.6
\end{aligned}$$

Inserting the first three equations in the sum-to-one constraint gives

$$\sum_{i=1}^{4} \pi_i = \pi_1(1 + 1 + \frac{0.5}{0.6} + \frac{0.5}{3 \cdot 0.6} = 1$$

$$\begin{aligned}
\pi_1 &= \frac{1}{1 + 1 + 0.83 + 0.83 \cdot 0.33} = 0.32 \\
\pi_2 &= 0.32 \\
\pi_3 &= 0.32 \cdot 0.83 = 0.27 \\
\pi_4 &= 0.32 \cdot 0.83 \cdot 0.33 = 0.09
\end{aligned}$$

$$P(X_{t-1}=4|X_t=1)=\frac{P(X_{t-1}=4,X_t=1)}{P(X_t=1)}$$

$$=\frac{P(X_{t-1}=4)P(X_t=1|X_{t-1}=4)}{P(X_t=1)}=\frac{0.09\cdot 0.6}{0.32}=0.16$$

c)

$$\lim_{t\to\infty} P(X_t=1,X_{t+1}=2,X_{t+2}=3,X_{t+3}=4)$$

$$=\pi_1 P(X_t=2|X_{t-1}=1)P(X_t=3|X_{t-1}=2)P(X_t=4|X_{t-1}=3)$$

$$=0.32\cdot 0.6\cdot 0.5\cdot 0.2=0.0193\approx 0.02$$

This means there will be about 2 events like this in every 100 days.

In both displays there are 200 days. For the top plot there are direct increasing events like this around time 55, 85, 105, 165. For the bottom plot they are around time 30, 65, 70, 125. This gives eight events in 400 days, which is what we would expect from the theoretical result.

d) $T=\min\{t; X_t=4\}$.

$u_i=E(T|X_0=i)$, $i=1,2,3$, $u_4=0$.

Using double expectation in a first step analysis, we have

$$u_i=\sum_{j=1}^{4}[E(T|X_1=j,X_0=i)+1]P(i,j)$$

$$\begin{aligned}
u_1 &= 1+u_1 0.4+u_2 0.6\\
u_2 &= 1+u_1 0.1+u_2 0.4+u_3 0.5\\
u_3 &= 1+u_1 0.4+u_3 0.4+0.2\cdot 0
\end{aligned}$$

$$\begin{aligned}
u_1 &= 1/0.6+u_2\\
u_2 &= 1+(1/0.6+u_2)0.1+u_2 0.4+u_3 0.5\\
u_3 &= 1+u_1 0.4+u_3 0.4+0.2\cdot 0
\end{aligned}$$

$$\begin{aligned}
u_1 &= 1.67 + u_2 \\
u_2 &= (1+0.167)/0.5 + u_3 \\
u_2 - (1+0.167)/0.5 &= 1 + (u_2 + 1.67)0.4 + u_2 0.4 - (1+0.167)/0.50.4 + 0.2 \cdot 0
\end{aligned}$$

We solve this for u_2 to get:

$$u_2 = \frac{(1+0.167)/0.5 + 1 + 1.67 \cdot 0.4 - (1+0.167)/0.5 \cdot 0.4}{1 - 0.4 - 0.4} = \frac{3.07}{0.2} = 15.33$$

$$u_1 = 1.67 + u_2 = 17.0$$

e) From Bayes theorem

$$P(X_t = k | y_t = 3.2) = \frac{P(X_t = k) p_{Y_t|X_t=k}(3.2|k)}{p_{Y_t}(3.2)} \propto \pi_k \exp\left(-\frac{(3.2-k)}{2}\right)$$

$$\begin{aligned}
\pi_1 \exp\left(-\frac{(3.2-1)^2}{2}\right) &= 0.32 \cdot 0.09 = 0.029 \\
\pi_2 \exp\left(-\frac{(3.2-2)^2}{2}\right) &= 0.32 \cdot 0.49 = 0.157 \\
\pi_3 \exp\left(-\frac{(3.2-3)^2}{2}\right) &= 0.27 \cdot 0.98 = 0.263 \\
\pi_4 \exp\left(-\frac{(3.2-4)^2}{2}\right) &= 0.09 \cdot 0.73 = 0.065
\end{aligned}$$

The probability of risk class 4 is:

$$P(X_t = 4 | y_t = 3.2) = \frac{0.065}{0.029 + 0.157 + 0.263 + 0.065} = 0.13$$

Similarly,

$$P(X_t = 3 | y_t = 3.2) = \frac{0.263}{0.029 + 0.157 + 0.263 + 0.065} = 0.51$$

By using independence between X_{t+1} and Y_t, given X_t, the probability at the next time $t+1$ is

$$P(X_{t+1}=4|y_t=3.2)=\sum_{k=1}^{4}P(X_t=k,X_{t+1}=4|y_t=3.2)$$

$$=\sum_{k=1}^{4}P(X_t=k|y_t=3.2)P(X_{t+1}=4|X_t=k,y_t=3.2)$$

$$=\sum_{k=1}^{4}P(X_t=k|y_t=3.2)P(X_{t+1}=4|X_t=k)$$

$$=P(X_t=3|y_t=3.2)P(3,4)+P(X_t=4|y_t=3.2)P(4,4)=0.51\cdot 0.2+0.13\cdot 0.4=0.15$$

Problem 2

a) $\lambda=0.5$.

$N(t)$ = number of customers in time $(0,t)$. This is Poisson distributed with parameter λt.

$$P(N(15)=0)=\exp(-\lambda 5)=\exp(-2.5)=0.08$$

$$E(N(15))=\lambda 15=7.5$$

$$P(N(5)=0|N(10)=2)=\frac{P(N(5)=0,N(10)-N(5)=2)}{P(N(10)=2)}$$

$$=\frac{\exp(-5\lambda)\cdot\frac{(\lambda 5)^2}{2}\exp(-5\lambda)}{\frac{(\lambda 10)^2}{2}\exp(-10\lambda)}=0.25$$

The interpretation is that two independent events occur, and since they are uniformly distributed within $(0,10)$, the chance that both occur within $(5,10)$ is $0.5^2=0.25$.

b) C_i is amount spent by customer i.

$$X=C_1+\ldots+C_{N(t)}$$

By double expectation:

$$E(X)=E(E(X|N(t)=n))=E(C)E(N(t))=100\cdot\lambda 60\cdot 8=24000$$

By double variance:

$$Var(X) = E(Var(X|N(t) = n)) + Var(E(X|N(t) = n)) = Var(C)E(N(t)) + E(C)^2 Var(N(t))$$

$$= 10^2 \cdot \lambda 60 \cdot 8 + 100^2 \cdot \lambda 60 \cdot 8 = (1557)^2$$

c) Arrivals are run by the Poisson process.

$$\lambda_i = \lambda$$

Departures are run by the Poisson process, but there are i customers that can leave, so the rate is $i\mu$.

$$\mu_i = i\mu$$

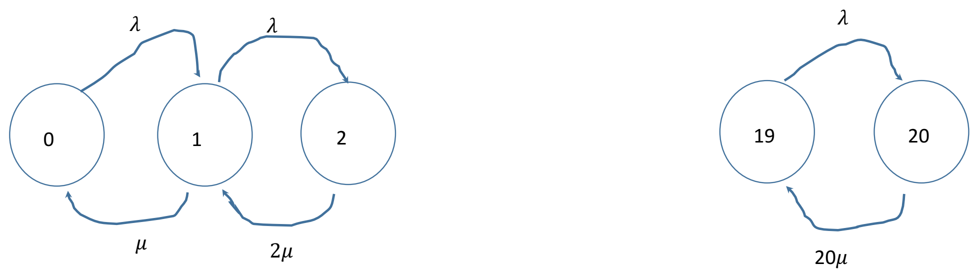

Figure 1: Transition diagram.

Define waiting time $T = T_2 + T_1$. T_2 is exponential distributed with parameter 2μ, while T_1 is exponential distributed with parameter μ. The density of T is

$$\begin{aligned} f_T(t) &= \int_0^t f_{T_2}(s) f_{T_1}(t-s) ds = \int_0^t 2\mu e^{-2\mu s} \mu e^{-\mu(t-s)} ds \\ &= e^{-\mu t} 2\mu \int_0^t \mu e^{-\mu s} ds = e^{-\mu t} 2\mu(-e^{-\mu t} + 1) \\ &= 2\mu e^{-\mu t} - 2\mu e^{-2\mu t} \end{aligned}$$

d) Long-run probabilities are $P_i = \lim_{t\to\infty} P(N(t) = i)$.

By equating the rates out and rates in we get

$$\begin{aligned} P_0\lambda &= P_1\mu \\ P_1(\lambda+\mu) &= P_0\lambda + P_2 2\mu \\ \dots &= \dots \\ P_{20}20\mu &= P_{19}\lambda \end{aligned}$$

$$P_i = \frac{\nu^i}{i!}P_0, \quad \nu = \lambda/\mu$$

$$P_0 = \frac{1}{\sum_{i=0}^{20}\frac{\nu^i}{i!}} = \frac{1}{\exp(\nu)F_X(20)}$$

Here, X is Poisson distributed with parameter $\nu = \lambda/\mu = 0.5/0.0333 = 15$. Then $\sum_{i=0}^{20}\frac{\nu^i}{i!} = \exp(\nu)F_X(20)$, where F_X is the cumulative distribution function.

This gives

$$P_{20} = \frac{f_X(20)}{F_X(20)} = \frac{(0.917 - 0.875)}{0.917} = 0.045$$

$$E(N) = \sum_{i=0}^{20} iP_i = P_0\sum_{i=0}^{20} i\frac{\nu^i}{i!} = P_0\nu\sum_{i=0}^{19}\frac{\nu^i}{i!}$$

$$\sum_{i=0}^{19}\frac{\nu^i}{i!} = \exp(\nu)\sum_{i=0}^{19}\exp(-\nu)\frac{\nu^i}{i!} = \exp(\nu)F_X(19)$$

$$E(N) = \frac{\nu\exp(\nu)F_X(19)}{\exp(\nu)F_X(20)} = \nu\frac{F_X(19)}{F_X(20)} = 15\frac{0.875}{0.917} = 14.31.$$

e) Assume today's cost per minute of parking is c_{min}. The expected long-run pay per minute is

For the same income we get:

$$E(N)c_{\text{min}} = E_{\text{new}}(N)2c_{\text{min}}$$

$$E_{\text{new}}(N) = \frac{E(N)}{2} = 7.155$$

Here, we let λ^* be the new arrival rate, which is used when computing the left hand side. Since one assumes no capacity at the parking garage, the new probabilities are approximated by

$$P_i = \frac{x^i}{i!} P_0, \quad x = \lambda^*/\mu, \quad i = 0, 1, 2, \ldots$$

$$P_0 = \frac{1}{\sum_{i=0}^{\infty} \frac{x^i}{i!}} = \frac{1}{\exp(x)} = \exp(-x)$$

These are the Poisson probabilities with parameter $x = \lambda^*/\mu$.

$$E_{\text{new}}(N) = \sum_{i=0}^{\infty} i P_i = \lambda^*/\mu, \lambda^* = \mu 7.155 = 0.0333 \cdot 7.155 = 0.238$$