NTNU
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Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Processes**

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Examination date: December 1, 2016

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottmann: Matematisk formelsamling.
- Bilingual dictionary.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers must be justified.
All ten subproblems are equally weighted.

Language: English

Number of pages: 4

Number of pages enclosed: 3

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1

There are risks of avalanche (masses of snow collapsing) several places in Norway. The collapse of a very large snow mass could mean a damaging avalanche. We here assume that the risk class (snow mass) is described by four categories: small (1), medium (2), large (3) and very large (4). During winter, we assume that this risk or snow mass will tend to be the same or grow, or the snow collapses. The dynamic risk class X_t, for day $t = 0, 1, 2, 3, ..$, is here modeled by a Markov chain with transition matrix:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.1 & 0.4 & 0.5 & 0 \\ 0.4 & 0 & 0.4 & 0.2 \\ 0.6 & 0 & 0 & 0.4 \end{pmatrix} \end{array},$$

where element $P(i, j) = P(X_t = j | X_{t-1} = i)$.

a)

Calculate $P(X_2 = 2, X_1 = 1 | X_0 = 1)$.

Calculate $P(X_2 = 2 | X_0 = 1)$.

Assume the risk is very large. What is the distribution for the time until collapse, and what is the expected time until it collapses?

b)

Calculate the long-run proportion for the different risk classes.

What is the probability that the risk was very large yesterday (time $t - 1$), given that it is small today (time t).

c)

Calculate the long-run proportion of time the risk process moves directly through the increasing risk classes, i.e. $X_t = 1$, $X_{t+1} = 2$, $X_{t+2} = 3$, $X_{t+3} = 4$.

Figure 1 shows realizations of this risk process from two winters. Compare the realized proportion of directly increasing risk classes with the theoretical results.

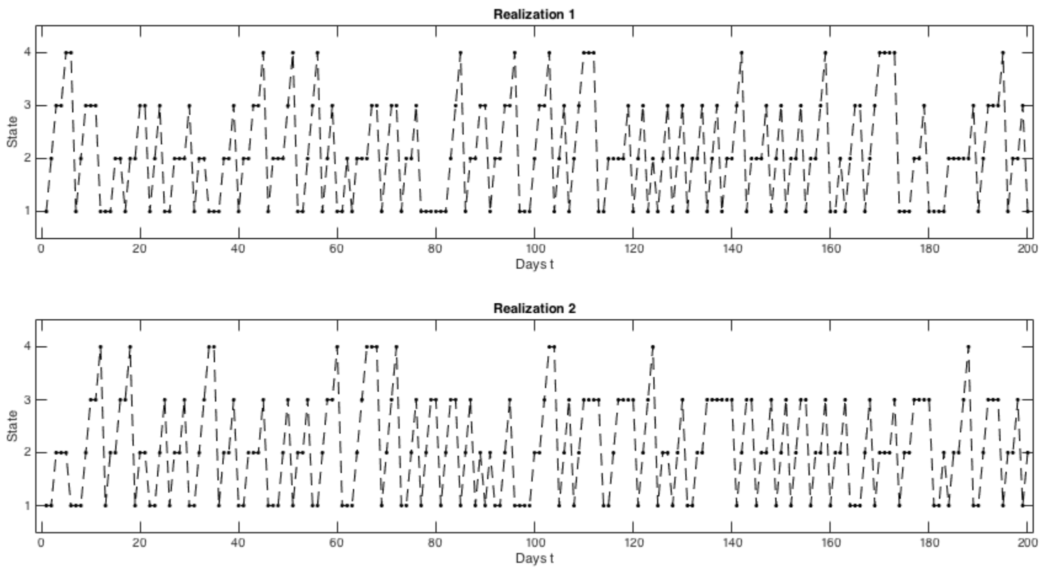

Figure 1: Two realizations of the process for avalanche risk.

d)

Assume $X_0 = 1$. Calculate the expected time until the risk class first reaches state 4.

e)

At a particular location of interest, along a railroad line, one can measure the risk with an automatic sensor. This measurement, denoted Y_t, will not give perfect information about the risk class X_t. It will be continuously distributed, and dependent on the risk only at the day the measurement is made. Given the risk class $X_t = k$, the measurement distribution is Gaussian, $\mathrm{N}(k, 1)$, i.e. the probability density function is

$$p(y_t|X_t = k) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(y_t - k)^2}{2}\right).$$

The measurement is $Y_t = 3.2$, what is the probability that the risk class is $X_t = 4$?

The measurement is $Y_t = 3.2$, what is the probability that the risk class at time $t + 1$ is very large?

Problem 2

At a parking garage, the arrivals of customers follow a Poisson process with rate $\lambda = 0.5$ (time t is in minutes).

a)

The garage opens at 8:00. What is the probability that no customer has arrived by 8:05?

What is the expected number of customers arriving the first 15 minutes?

Given that two customers arrived during the first 10 minutes, what is the probability that no customer arrived the first 5 minutes?

b)

Customers independently spend on average 100 kr for parking, with standard deviation 10 kr.

Calculate the expected value of the total income at the garage during the day (08:00-16:00).

Calculate the variance of the total income at the garage during the day (08:00-16:00).

Consider next the more realistic situation where customers arrive and leave the parking garage. We assume arrivals are independent according to the Poisson process described above. We further assume customers independently spend an exponential distributed time in the parking garage, with expectation $1/\mu$. We set $\mu = 1/30 = 0.0333$. The number of customers $N(t)$ in the garage by time t can then be modeled by a birth-and-death process. The maximum capacity of the garage is $N_{\text{max}} = 20$. If there are 20 in the garage, arriving customers will just drive by the garage, without forming a queue.

c)

Find the birth and death rates of the process.

Draw a transition diagram for the process.

The garage is closing. They receive no more arriving customers, while the customers who are in the garage leave at the rates described above. There are two customers at closing time. Find the probability density of the waiting time until the parking garage is empty.

d)

Use long-run equality of process rates going in and out of states to find expressions for the long-run probabilities of the process. Compute $P_{20} = \lim_{t\to\infty} P(N(t) = 20)$.

Compute the expected long-run number of customers in the garage; $E(N)$.

(Hint: You can look up Poisson probabilities in a table. If X is Poisson distributed with parameter ν, we have $P(X = i) = e^{-\nu}\frac{\nu^i}{i!}$, $i = 0, 1, 2, \ldots$.)

e)

For environmental reasons the authorities demand that garages must double the pay per minute for parking. The owner of a garage thinks this will reduce the arrival rate of costumers. One assumes that the customers on average spend the same time as described above in the garage. One further assumes that the parking capacity will never be reached in this case.

Find the new arrival rate that makes the expected long-term income stay the same as before. (Take for granted an expectation $E(N)$ as the solution to exercise 2d.)

Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$
$$P_{i,i-1}(h) = \mu_i h + o(h)$$
$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$
$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i \,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij} \,, \; i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1-a)^2} \quad .$$