NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4265 Stochastic Processes**

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Examination date: December 14th, 2015

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Calculator CITIZEN SR-270X, CITIZEN SR-270X College, HP30S, Casio fx-82ES PLUS with empty memory.
- Tabeller og formler i statistikk, Tapir forlag.
- K. Rottmann: Matematisk formelsamling.
- Bilingual dictionary.
- One yellow, stamped A5 sheet with own handwritten formulas and notes (on both sides).

Other information:
Note that all answers must be justified.
In your solution you can use English and/or Norwegian.
All ten subproblems are approximately equally weighted.

Language: English

Number of pages: 3

Number of pages enclosed: 3

Checked by:

Date Signature

Problem 1

On any given day Gary is either cheerful (0), so-so (1), or glum (2). Let X_n denote Gary's mood on the nth day and assume $\{X_n, n \geq 0\}$ is a Markov chain with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.4 & x & 0 \\ 0.3 & 0.6 & y \\ 0 & 0.45 & z \end{pmatrix} \end{array}$$

a)
- Find x, y and z.
- Derive $P(X_3 = 0 \mid X_2 = 0, X_1 \neq 0, X_0 = 0)$.
- Assume Gary's mood is "so-so". What is the probability that when Gary changes his mood, he changes to "glum" rather than "cheerful"?

b) Determine the long-run proportion of time that Gary is in a glum mood.

Problem 2

A coin is flipped repeatedly and independently. The probability to get head (H) is denoted by p while the probability to get tail (T) is $q = 1 - p$.

a) Use a Markov chain formulation and first-step analysis to derive the expected number of flips made until the pattern H, T, H appears for the first time.

Problem 3

a) Give an example of a discrete-time Markov chain with state space $S = \{0, 1, 2\}$, that is irreducible and ergodic but **not** time-reversible. Explain why your example is a valid choice, i.e. why it fulfills all mentioned criteria.

Problem 4

David goes fishing and catches fish according to a Poisson process with rate 2 per hour.

a)
- What is the probability that he catches at least 2 fishes within the first half an hour?
- Assume David caught exactly 3 fishes in the first hour, when is he expected to catch the 9th fish?

Suppose further that each catch is a salmon with probability $p = 0.4$, and a trout with probability $q = 1 - p = 0.6$. Whether a catch is a salmon or a trout is independent of all other catches.

b) Derive the probability that David will catch exactly 1 salmon and 2 trouts in a given 2 hour and 30 minute period.

Assume David is fishing together with two friends. Each of them also catches fish (independently of the others) according to a Poisson process with rate 2.

c) What is the expected time until everyone has caught at least one fish?

Problem 5

A tourist guide is stationed at a port and hired to give sightseeing tours with his boat. If the guide is free, it takes an interested tourist a certain time, exponentially distributed with mean $1/\mu_1$, to negotiate the price and type of tour. Assume there is a probability α that there is no agreement and the tourist leaves. If the tourist and the guide reach an agreement, they start directly the tour. The duration of the tour is exponentially distributed with mean $1/\mu_2$. Suppose that interested tourists arrive at the guide's station according to a Poisson process with rate λ and request service only when the guide is free, i.e. the guide is neither negotiating nor on tour with another tourist. Suppose further that the Poisson process, the negotiation time, the duration of the tour and whether the tourist and the guide achieve an agreement are independent.

a) Set up the set of balance equations and derive the proportion of time (in the long run) that the guide is free.

b) Find the transition probability matrix of the embedded discrete-time Markov chain. Determine the period for all states of this Markov chain.

Problem 6

Assume you are able to only simulate standard normal distributed random variables.

a) Give an algorithm for simulating a Brownian motion process with drift coefficient μ and variance parameter σ^2 at times $0 = t_0 < t_1 < t_2 < \cdots < t_k$.

Formulas for TMA4265 Stochastic Processes:

The law of total probability

Let $B_1, B_2, \ldots$ be pairwise disjoint events with $P(\cup_{i=1}^{\infty} B_i) = 1$. Then

$$P(A|C) = \sum_{i=1}^{\infty} P(A|B_i \cap C)P(B_i|C),$$

$$E[X|C] = \sum_{i=1}^{\infty} E[X|B_i \cap C]P(B_i|C).$$

Discrete time Markov chains

Chapman-Kolmogorov equations

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)}.$$

For an irreducible and ergodic Markov chain, $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)}$ exist and is given by the equations

$$\pi_j = \sum_i \pi_i P_{ij} \quad \text{and} \quad \sum_i \pi_i = 1.$$

For transient states i, j and k, the expected time spent in state j given start in state i, s_{ij}, is

$$s_{ij} = \delta_{ij} + \sum_k P_{ik} s_{kj}.$$

For transient states i and j, the probability of ever returning to state j given start in state i, f_{ij}, is

$$f_{ij} = (s_{ij} - \delta_{ij})/s_{jj}.$$

The Poisson process

The waiting time to the n-th event (the n-th arrival time), S_n, has the probability density

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} e^{-\lambda t} \quad \text{for} \quad t \geq 0.$$

Given that the number of events $N(t) = n$, the arrival times $S_1, S_2, \ldots, S_n$ have the joint probability density

$$f_{S_1,S_2,\ldots,S_n|N(t)}(s_1, s_2, \ldots, s_n|n) = \frac{n!}{t^n} \quad \text{for} \quad 0 < s_1 < s_2 < \ldots < s_n \leq t.$$

Markov processes in continuous time

A (homogeneous) Markov process $X(t)$, $0 \leq t \leq \infty$, with state space $\Omega \subseteq \mathbf{Z}^+ = \{0, 1, 2, \ldots\}$, is called a birth and death process if

$$P_{i,i+1}(h) = \lambda_i h + o(h)$$
$$P_{i,i-1}(h) = \mu_i h + o(h)$$
$$P_{i,i}(h) = 1 - (\lambda_i + \mu_i)h + o(h)$$
$$P_{ij}(h) = o(h) \quad \text{for } |j - i| \geq 2$$

where $P_{ij}(s) = P(X(t+s) = j | X(t) = i)$, $i, j \in \mathbf{Z}^+$, $\lambda_i \geq 0$ are birth rates, $\mu_i \geq 0$ are death rates.

The Chapman-Kolmogorov equations

$$P_{ij}(t+s) = \sum_{k=0}^{\infty} P_{ik}(t) P_{kj}(s).$$

Limit relations

$$\lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = v_i\,, \quad \lim_{h \to 0} \frac{P_{ij}(h)}{h} = q_{ij}\,,\ i \neq j$$

Kolmogorov's forward equations

$$P'_{ij}(t) = \sum_{k \neq j} q_{kj} P_{ik}(t) - v_j P_{ij}(t).$$

Kolmogorov's backward equations

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t).$$

If $P_j = \lim_{t \to \infty} P_{ij}(t)$ exist, P_j are given by

$$v_j P_j = \sum_{k \neq j} q_{kj} P_k \quad \text{and} \quad \sum_j P_j = 1.$$

In particular, for birth and death processes

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \theta_k} \quad \text{and} \quad P_k = \theta_k P_0 \quad \text{for} \quad k = 1, 2, \ldots$$

where

$$\theta_0 = 1 \quad \text{and} \quad \theta_k = \frac{\lambda_0 \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \mu_2 \cdot \ldots \cdot \mu_k} \quad \text{for} \quad k = 1, 2, \ldots$$

Queueing theory

For the average number of customers in the system L, in the queue L_Q; the average amount of time a customer spends in the system W, in the queue W_Q; the service time S; the average remaining time (or work) in the system V, and the arrival rate λ_a, the following relations obtain

$$L = \lambda_a W.$$

$$L_Q = \lambda_a W_Q.$$

$$V = \lambda_a E[SW_Q^*] + \lambda_a E[S^2]/2.$$

Some mathematical series

$$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a} \quad , \qquad \sum_{k=0}^{\infty} k a^k = \frac{a}{(1 - a)^2} \quad .$$