Plan for this lecture

- Recall some theory parts
  - additive regression models
    - situation
    - notation
    - estimators
    - TST test
  - some formulas found in the example (defined below)
  - TST test
- Additive regression model with one binary covariate
  - Problem 4.4 in ABG
  - Problem 4.5 a) in ABG

Additive regression models — situation

- n individuals
- individual i has covariate vector $x_i(t)$
- individual i has hazard rate and intensity process

$$\begin{aligned} \alpha(t|x_i(t)) &= \alpha_0(t) r(\beta, x_i(t)) \\ \lambda_i(t) &= Y_i(t)\alpha_0(t) r(\beta, x_i(t)) \end{aligned}$$

where

$$r(\beta, x_i(t)) = \beta_0(t) + \beta_1(t)x_{i1}(t) + \ldots \beta_p(t)x_{ip}(t)$$

- $N_i(t)$, $Y_i(t)$, $N_\bullet(t)$, $Y_\bullet(t)$, $\lambda_\bullet(t)$

⋆ $B_q(t) = \int_0^t \beta_q(u)du$ for $q = 0, 1, \ldots, p$

Additive regression models — notation and estimator

- ⋆ $\mathbb{N}(u) = [N_1(u) N_2(u) \cdots N_n(u)]^T$
- ⋆ $\mathbb{X}(u)$:

$$\mathbb{X}(u) = \begin{bmatrix} Y_1(u) & Y_1(u)x_{11}(u) & \cdots & Y_1(u)x_{1p}(u) \\ Y_2(u) & Y_2(u)x_{21}(u) & \cdots & Y_2(u)x_{2p}(u) \\ \vdots & \vdots & & \vdots \\ Y_n(u) & Y_n(u)x_{n1}(u) & \cdots & Y_n(u)x_{np}(u) \end{bmatrix}$$

- ⋆ $\mathbb{X}^-(u) = (\mathbb{X}(u)^T \mathbb{X}(u))^{-1} \mathbb{X}(u)^T$, $(p+1) \times n$ matrix
- ⋆ $J(u) = I(\mathbb{X}(u)$ has full rank), scalar
- ⋆ Estimator

$$\widehat{\mathbb{B}}(t) = \begin{bmatrix} \widehat{B}_0(t) \\ \widehat{B}_1(t) \\ \vdots \\ \widehat{B}_p(t) \end{bmatrix} = \int_0^t J(u) \mathbb{X}^-(u) d\mathbb{N}(u)$$

Stochastic integral with respect to a vector process

- Let $\mathbb{H}(u)$ be a $p \times k$ matrix of predictable processes. Then we define

$$\int_0^t \mathbb{H}(u) d\mathbb{M}(u)$$

to be a p-vector where element h is

$$\left[\int_0^t \mathbb{H}(u) d\mathbb{M}(u)\right]_h = \sum_{j=1}^{k} \int_0^t H_{hj}(u) dM_j(u)$$

Some results in our example situation

- ⋆ $\mathbb{X}(u)$:

$$\mathbb{X}(u) = \begin{bmatrix} Y_1(u) & Y_1(u)x_1 \\ Y_2(u) & Y_2(u)x_2 \\ \vdots & \vdots \\ Y_n(u) & Y_n(u)x_n \end{bmatrix}$$

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- ⋆ $\mathbb{X}(u)^T\mathbb{X}(u)$:

$$\mathbb{X}(u)^T\mathbb{X}(u) = \begin{bmatrix} Y_\bullet(u) & Y^{(1)}(u) \\ Y^{(1)}(u) & Y^{(1)}(u) \end{bmatrix}$$

Some results in our example situation

- ⋆ $\mathbb{X}(u)$:

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- ⋆ $\mathbb{X}(u)^T\mathbb{X}(u)$:

$$\mathbb{X}(u)^T\mathbb{X}(u) = \begin{bmatrix} Y_\bullet(u) & Y^{(1)}(u) \\ Y^{(1)}(u) & Y^{(1)}(u) \end{bmatrix}$$

- ⋆ $\left(\mathbb{X}(u)^T\mathbb{X}(u)\right)^{-1}$:

$$\left(\mathbb{X}(u)^T\mathbb{X}(u)\right)^{-1} = \begin{bmatrix} \frac{1}{Y^{(\mathbf{0})}(u)} & -\frac{1}{Y^{(\mathbf{0})}(u)} \\ -\frac{1}{Y^{(\mathbf{0})}(u)} & \frac{1}{Y^{(\mathbf{0})}(u)} + \frac{1}{Y^{(\mathbf{1})}(u)} \end{bmatrix}$$

Some results in our example situation

- $\mathbb{X}(u)$:

$$\mathbb{X}(u) = \begin{bmatrix} Y_1(u) & Y_1(u)x_1 \\ Y_2(u) & Y_2(u)x_2 \\ \vdots & \vdots \\ Y_n(u) & Y_n(u)x_n \end{bmatrix}$$

- $\mathbb{X}(u)^T\mathbb{X}(u)$:

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- $\left(\mathbb{X}(u)^T\mathbb{X}(u)\right)^{-1}$:

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$$\mathbb{X}^{-}(u) = \begin{bmatrix} \frac{Y_\mathbf{1}(u)(1-x_\mathbf{1})}{Y^{(\mathbf{0})}(u)} & \frac{Y_\mathbf{2}(u)(1-x_\mathbf{2})}{Y^{(\mathbf{0})}(u)} & \cdots & \frac{Y_n(u)(1-x_n)}{Y^{(\mathbf{0})}(u)} \\ \frac{Y_\mathbf{1}(u)x_\mathbf{1}}{Y^{(\mathbf{1})}(u)} - \frac{Y_\mathbf{1}(u)(1-x_\mathbf{1})}{Y^{(\mathbf{0})}(u)} & \frac{Y_\mathbf{2}(u)x_\mathbf{2}}{Y^{(\mathbf{1})}(u)} - \frac{Y_\mathbf{2}(u)(1-x_\mathbf{2})}{Y^{(\mathbf{0})}(u)} & \cdots & \frac{Y_n(u)x_n}{Y^{(\mathbf{1})}(u)} - \frac{Y_n(u)(1-x_n)}{Y^{(\mathbf{0})}(u)} \end{bmatrix}$$

TST test

- ⋆ Additive regression regression situation
- ⋆ Want to test $H_0 : \beta_q(t) = 0$ for $t \in [0, t_0]$
- ⋆ Can use test statistic

$$\frac{Z_q(t_0)}{\sqrt{V_{qq}(t_0}}$$

which is approximately standard normal when H_0 is true

- ⋆ $Z_q(t_0)$ and $V_{qq}(t_0)$ is given by

$$Z_q(t_0) = \int_0^{t_0} L_q(t) d\widehat{B}_q(t) = \sum_{j: T_j \le t_0} L_q(T_j) \Delta \widehat{B}_q(T_j)$$

$$V_{qq}(t_0) = \int_0^{t_0} L_q^2(t) d\widehat{\sigma}_{qq}(t) = \sum_{j: T_j \le t_0} L_q^2(T_j) \Delta \widehat{\sigma}_{qq}(t)$$

and

$$L_q(t) = \mathbb{K}_{qq}(t),$$

$$\mathbb{K}(t) = \left(\text{diag} \left[\left(\mathbb{X}(t)^T \mathbb{X}(t) \right)^{-1} \right] \right)^{-1}$$