Plan for this lecture

* Summary of nonparametric tests for two groups of individuals
  - test for two groups, $H_0 : \alpha_1(t) = \alpha_2(t)$
* Nonparametric test with only one group av individuals

$$H_0 : \alpha(t) = \alpha_0(t)$$

Nonparametric test for two groups

- Two counting processes: $N_1(t)$ and $N_2(t)$

$$\lambda_1(t) = \alpha_1(t) Y_1(t) \quad \text{and} \quad \lambda_2(t) = \alpha_2(t) Y_2(t)$$

- Want to test $H_0 : \alpha_1(t) = \alpha_2(t)$ for $t \in [0, t_0]$
- Consider statistic

$$\begin{aligned} Z_1(t_0) &= \int_0^{t_0} L(t)(d\widehat{A}_1(t) - d\widehat{A}_2(t)) \\ &= \int_0^{t_0} \frac{L(t)}{Y_1(t)} dN_1(t) - \int_0^{t_0} \frac{L(t)}{Y_2(t)} dN_2(t) \end{aligned}$$

Nonparametric test for two groups

- ⋆ Two counting processes: $N_1(t)$ and $N_2(t)$

$$\lambda_1(t) = \alpha_1(t)Y_1(t) \quad \text{and} \quad \lambda_2(t) = \alpha_2(t)Y_2(t)$$

- ⋆ Want to test $H_0 : \alpha_1(t) = \alpha_2(t)$ for $t \in [0, t_0]$
- ⋆ Consider statistic

$$\begin{aligned} Z_1(t_0) &= \int_0^{t_0} L(t)(d\widehat{A}_1(t) - d\widehat{A}_2(t)) \\ &= \int_0^{t_0} \frac{L(t)}{Y_1(t)} dN_1(t) - \int_0^{t_0} \frac{L(t)}{Y_2(t)} dN_2(t) \end{aligned}$$

- ⋆ Use Doob-Meyer decompositions of $N_1(t)$ and $N_2(t)$ and get

$$Z_1(t_0) = \int_0^{t_0} L(t)(\alpha_1(t) - \alpha_2(t))dt + \int_0^{t_0} \frac{L(t)}{Y_1(t)} dM_1(t) - \int_0^{t_0} \frac{L(t)}{Y_2(t)} dM_2(t)$$

Nonparametric test for two groups

- Two counting processes: $N_1(t)$ and $N_2(t)$

$$\lambda_1(t) = \alpha_1(t)Y_1(t) \quad \text{and} \quad \lambda_2(t) = \alpha_2(t)Y_2(t)$$

- Want to test $H_0 : \alpha_1(t) = \alpha_2(t)$ for $t \in [0, t_0]$
- Consider statistic

$$\begin{aligned} Z_1(t_0) &= \int_0^{t_0} L(t)(d\widehat{A}_1(t) - d\widehat{A}_2(t)) \\ &= \int_0^{t_0} \frac{L(t)}{Y_1(t)} dN_1(t) - \int_0^{t_0} \frac{L(t)}{Y_2(t)} dN_2(t) \end{aligned}$$

- Use Doob-Meyer decompositions of $N_1(t)$ and $N_2(t)$ and get

$$Z_1(t_0) = \int_0^{t_0} L(t)(\alpha_1(t) - \alpha_2(t))dt + \int_0^{t_0} \frac{L(t)}{Y_1(t)} dM_1(t) - \int_0^{t_0} \frac{L(t)}{Y_2(t)} dM_2(t)$$

- When H_0 is true $Z(t_0)$ is a mean zero martingale

Properties of $Z(t_0)$ when H_0 is true

- $\mathrm{E}[Z(t_0)] = 0$
- Using the predictable variation process $\langle Z \rangle(t_0)$ we found

$$\mathrm{Var}[Z(t_0)] = \mathrm{E}\left[\int_0^{t_0} \frac{L^2(t)(Y_1(t) + Y_2(t))}{Y_1(t)Y_2(t)}\alpha(t)dt\right]$$

- Estimating $\alpha(t)dt$ with $d\widehat{A}(t)$ (data from both groups) we get an estimator for $\mathrm{Var}[Z(t_0)]$

$$V_{11}(t_0) = \int_0^{t_0} \frac{L^2(t)}{Y_1(t)Y_2(t)}dN_\bullet(t)$$

- It can be shown (Chapter 3.3.5 in ABG) that $Z(t_0)$ is approximately normal
- Use test statistic

$$U(t_0) = \frac{Z(t_0)}{\sqrt{V_{11}(t_0)}}$$

  which is approximately standard normal when H_0 is true

One sample nonparametric test

- Assume:
  - n individuals
  - each individual has the same hazard rate $\alpha(t)$
  - no tied observations
  - $N(t)$: # individuals failed up to (and including) time t
  - $Y(t)$: # individuals at risk just before time t
- Multiplicative intensity model

$$\lambda(t) = \alpha(t) Y(t)$$

One sample nonparametric test

- ⋆ Assume:
  - n individuals
  - each individual has the same hazard rate $\alpha(t)$
  - no tied observations
  - $N(t)$: # individuals failed up to (and including) time t
  - $Y(t)$: # individuals at risk just before time t
- ⋆ Multiplicative intensity model

$$\lambda(t) = \alpha(t)Y(t)$$

- ⋆ Want to test

$$H_0 : \alpha(t) = \alpha_0(t) \quad \text{for } t \in [0, t_0]$$

where $\alpha_0(t)$ is a known hazard rate

One sample nonparametric test

- Assume:
  - n individuals
  - each individual has the same hazard rate $\alpha(t)$
  - no tied observations
  - $N(t)$: # individuals failed up to (and including) time t
  - $Y(t)$: # individuals at risk just before time t
- Multiplicative intensity model

$$\lambda(t) = \alpha(t)Y(t)$$

- Want to test

$$H_0 : \alpha(t) = \alpha_0(t) \quad \text{for } t \in [0, t_0]$$

  where $\alpha_0(t)$ is a known hazard rate
- The test statistic should be based on

$$\widehat{A}(t) = \int_0^t \frac{J(s)}{Y(s)} dN(s)$$

One sample nonparametric test

- ⋆ Assume:
  - n individuals
  - each individual has the same hazard rate $\alpha(t)$
  - no tied observations
  - $N(t)$: # individuals failed up to (and including) time t
  - $Y(t)$: # individuals at risk just before time t
- ⋆ Multiplicative intensity model

$$\lambda(t) = \alpha(t)Y(t)$$

- ⋆ Want to test

$$H_0 : \alpha(t) = \alpha_0(t) \quad \text{for } t \in [0, t_0]$$

where $\alpha_0(t)$ is a known hazard rate

- ⋆ The test statistic should be based on

$$\widehat{A}(t) = \int_0^t \frac{J(s)}{Y(s)} dN(s)$$

- ⋆ Define

$$A_0^\star(t) = \int_0^t J(s)\alpha_0(s) ds$$

What we are going to do — step by step

- $\star$ Show that $\widehat{A}(t) - A_0^\star(t)$ is a mean zero martingale (under H_0)
- $\star$ Find predictable variation process for $\widehat{A}(t) - A_0^\star(t)$ (under H_0)
- $\star$ Define statistic $Z(t_0)$
  - including a weight function $L(t)$
- $\star$ Find mean and variance of $Z(t_0)$ (under H_0)
  - define unbiased estimator for $\mathrm{Var}[Z(t_0)]$
- $\star$ Argue that $Z(t_0)$ is approximately normal (under H_0)
- $\star$ Consider the weight function $L(t) = Y(t)$

What we are going to do — step by step

- ⋆ Show that $\widehat{A}(t) - A_0^\star(t)$ is a mean zero martingale (under H_0)
- ⋆ Find predictable variation process for $\widehat{A}(t) - A_0^\star(t)$ (under H_0)
- ⋆ Define statistic $Z(t_0)$
  - including a weight function $L(t)$
- ⋆ Find mean and variance of $Z(t_0)$ (under H_0)
  - define unbiased estimator for $\mathrm{Var}[Z(t_0)]$
- ⋆ Argue that $Z(t_0)$ is approximately normal (under H_0)
- ⋆ Consider the weight function $L(t) = Y(t)$

- ⋆ First: Review some martingale results we are going to use

Martingale results

- ⋆ Doob-Meyer decomposition of a counting process $N(t)$

$$N(t) = \int_0^t \lambda(s)ds + M(t)$$

- ⋆ Predictable variation process for a counting process martingale

$$\langle M \rangle(t) = \int_0^t \lambda(s)ds$$

- ⋆ A stochastic integral with respect to a mean zero martingale is a mean zero martingale

$$I(t) = \int_0^t H(t)dM(t)$$

- ⋆ Predictable variation process for a stochastic integral

$$\langle I \rangle(t) = \int_0^t H^2(t)d\langle M \rangle(t)$$

- ⋆ Martingale central limit theorem