Plan for the exercise

- ⋆ Homogeneous Poisson process: Show $\langle M \rangle(t) = \lambda t$
  - Doob-Meyer decomposition: $N(t) = \lambda t + M(t)$
  - show that $M^2(t) - \lambda t$ is a mean zero martingale
  - show that the compensator of $M^2(t)$ is λt
  - show that the compensator of $M^2(t)$ is $\langle M \rangle(t)$
  - conclude that $\langle M \rangle(t) = \lambda t$
- ⋆ Poisson process
  - how to simulate
  - how to represent in a computer
  - how to plot $N(t)$, $N^\star(t)$ and $M(t)$

Continuous time martingale

- Martingale property:

$$\mathrm{E}[M(t)|\mathcal{F}_s] = M(s) \quad \text{for } s < t$$

- Consequences of the martingale property
  - constant mean

$$\mathrm{E}[M(t)] = \mathrm{E}[M(0)]$$

  - uncorrelated increments

$$\mathrm{Cov}[M(t)-M(s), M(v)-M(u)] = 0 \quad \text{for } 0 \leq s < t \leq u < v \leq \tau$$

Variation processes

- ⋆ Predictable variation process

$$\langle M \rangle(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Var}[M_k - M_{k-1} | \mathcal{F}_{\frac{(k-1)t}{n}}]$$

  - informally: $d\langle M \rangle(t) = \mathrm{Var}[dM(t) | \mathcal{F}_{t-}]$

- ⋆ Optional variation process

$$[M](t) = \lim_{n\to\infty} \sum_{k=1}^{n} (M_k - M_{k-1})^2$$

- ⋆ Consequences av the definitions
  - $M^2 - \langle M \rangle$ is a mean zero martingale
  - $M^2 - [M]$ is a mean zero martingale

Covariation processes

- ⋆ Predictable covariation process

$$\langle M_1, M_2\rangle(t) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{Cov}[\Delta M_{1k}, \Delta M_{2k} | \mathcal{F}_{\frac{(k-1)t}{n}}]$$

- ⋆ Optional covariation process

$$[M_1, M_2](t) = \lim_{n\to\infty} \sum_{k=1}^{n} (\Delta M_{1k})(\Delta M_{k2})$$

- ⋆ Consequences av the definitions
  - $M_1 M_2 - \langle M_1, M_2\rangle$ is a mean zero martingale
  - $M_1 M_2 - [M_1, M_2]$ is a mean zero martingale
  - $\langle aM_1, M_2\rangle = a\langle M_1, M_2\rangle$ and so on
  - $[aM_1, M_2] = a[M_1, M_2]$ and so on
  - $\langle M, M\rangle = \langle M\rangle$ and $[M, M] = [M]$

Stochastic integral

- Stochastic integral

$$I(t) = \int_0^t H(s) dM(s) = \lim_{n\to\infty} \sum_{k=1}^{n} H_k(M_k - M_{k-1})$$

- Consequences of the definitions
  - $I(t)$ is a mean zero martingale
  - $\langle \int HdM \rangle = \int H^2 d\langle M \rangle$
  - $[\int HdM] = \int H^2 d[M]$
  - $\langle \int H_1 dM_1, \int H_2 dM_2 \rangle = \int H_1 H_2 d\langle M_1, M_2 \rangle$
  - $[\int H_1 dM_1, \int H_2 dM_2] = \int H_1 H_2 d[M_1, M_2]$

Doob-Meyer decomposition

- Sub-martingale: X is a sub-martingale if

$$\mathrm{E}[X(t)|\mathcal{F}_s] \geq X(s) \quad \text{for } s < t$$

  - i.e. $X(t)$ tends to increase

- Doob-Meyer: Assume X is a sub-martingale with respect to $\{\mathcal{F}_t\}$, where $X(0) = 0$. Then there exists a unique decomposition

$$X = X^\star + M$$

  where

  - $X^\star$ is non-decreasing and predictable (the compensator for X)
  - M is a zero-mean martingale

- Informally:

  - $dX^\star(t) = \mathrm{E}[dX(t)|\mathcal{F}_{t-}]$ (predictable from the past)
  - $dM(t) = dX(t) - \mathrm{E}[dX(t)|\mathcal{F}_{t-}]$ (inovation/surprise)

The Poisson process

- $N(t)$: number of events in $[0, t]$ in a homogeneous Poisson process with intensity λ
- Doob-Meyer decomposition

$$N(t) = N^{\star}(t) + M(t)$$

where

  - $N^{\star}(t) = \lambda t$
  - $M(t) = N(t) - \lambda t$

Counting processes

- ⋆ $N(t)$: A counting process, $N(t)$ adapted to $\{\mathcal{F}_t\}$
  - right continuous
  - jumps of size one
  - constant between jumps
- ⋆ Recall informal definition of intensity process

$$\lambda(t)dt = P(dN(t) = 1|\mathcal{F}_{t-}) = \mathrm{E}[dN(t)|\mathcal{F}_{t-}]$$

- ⋆ Precise definition of intensity process by Doob-Meyer

$$N(t) = \Lambda(t) + M(t)$$

- ⋆ When $\Lambda(t)$ is absolutely continuous

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

- ⋆ Variation processes

$$\langle M\rangle(t) = \Lambda(t) = \int_0^t \lambda(s)ds \quad \text{and} \quad [M](t) = N(t)$$