Plan for the exercise

- ⋆ Show equivalence of two definitions of martingales
- ⋆ Partial sums of zero-mean independent stochastic variables
  - show it is a zero-mean martingale
  - find predictable variation process
  - find optional variation process
- ⋆ Partial sums of conditionally independent stochastic variables

Discrete time martingale

- ⋆ Martingale property:

$$E[M_n|\mathcal{F}_{n-1}] = M_{n-1}, n = 1, 2, \ldots$$

  - equivalently

$$E[M_n|\mathcal{F}_m] = M_m, n > m$$

- ⋆ Consequences of the martingale property
  - constant mean

$$E[M_n] = E[M_0], n = 1, 2, \ldots$$

  - uncorrelated increments

$$\mathrm{Cov}[M_m, M_n - M_m] = 0, n > m$$

Variation processes

- Predictable variation process

$$\langle M \rangle_n = \sum_{i=1}^{n} \mathrm{Var}[M_i - M_{i-1} | \mathcal{F}_{i-1}] = \sum_{i=1}^{n} \mathrm{E}\left[(M_i - M_{i-1})^2 \middle| \mathcal{F}_{i-1} \right]$$

- Optional variation process

$$[M]_n = \sum_{i=1}^{n} (M_i - M_{i-1})^2$$

- Consequences av the definitions
  - $M^2 - \langle M \rangle$ is a mean zero martingale
  - $M^2 - [M]$ is a mean zero martingale

Stopping times and transformations

- Stopping time T: The event $\{T = t\}$ is only dependent on what happens up (including) to time t
  - stopped process M^T:

$$M_n^T = M_{\min\{n,T\}}$$

- H is predictable based on $\{\mathcal{F}_n\}$ if H_n is known based on $\mathcal{F}_{n-1}$.
- Transformation of X by H, $Z = H \bullet X$

$$Z_n = H_0 X_0 + H_1(X_1 - X_0) + \ldots + H_n(X_n - X_{n-1})$$

- Consequences of the definitions
  - if M mean zero martingale, $H \bullet M$ mean zero martingale
  - $\langle H \bullet M \rangle = H^2 \bullet \langle M \rangle$
  - $[H \bullet M] = H^2 \bullet [M]$

Doob decomposition

- ⋆ Assume a process X with respect to $\{\mathcal{F}_n\}$, where $X_0 = 0$
- ⋆ Define M and $X^\star$

$$M_0 = X_0, \Delta M_n = M_n - M_{n-1} = X_n - \mathsf{E}[X_n|\mathcal{F}_{n-1}], n \geq 1$$

$$X_0^\star = 0, X_n^\star = \mathsf{E}[X_n|\mathcal{F}_{n-1}], n \geq 1$$

- ⋆ Then we have
  - the Doob decomposition

$$X_n = X_n^\star + \Delta M_n$$

  - M is a mean zero martingale
  - $X^\star$ is predictable with respect to $\{\mathcal{F}_n\}$