Plan for the exercise

- Nelson-Aalen for an aggregated Poisson process
  - show that it has a multiplicative intensity process
  - write down the Nelson-Aalen estimator
  - in a simple example: compute and plot the Nelson-Aalen estimator and confidence intervals
- When we have two populations
  - how to evaluate graphically whether the two hazard rates seem to be proportional?

Multiplicative intensity model

- Multiplicative intensity model

$$\lambda(t) = \alpha(t)Y(t)$$

  - $Y(t)$: predictable process

- Frequent situation leading to a multiplicative intensity model
  - n individuals
  - each individual has the same hazard rate $\alpha(t)$
  - may have truncation and/or censoring
  - $Y(t)$: number of individuals at risk just before time t
- Are two hazard rates proportional?

Nelson–Aalen estimator

- $N(t)$: counting process with $\lambda(t) = \alpha(t)Y(t)$
- Notation:
  - $J(t) = \mathbb{I}(Y(t) > 0)$
  - $A(t) = \int_0^t \alpha(t)dt$
  - $A^\star(t) = \int_0^t J(t)\alpha(t)dt$
- Nelson–Aalen estimator for $A(t)$:

$$\widehat{A}(t) = \int_0^t \frac{J(s)}{Y(s)} dN(s) = \sum_{j: T_j \le t} \frac{1}{Y(T_j)}$$

- Recall: $E[\widehat{A}(t) - A^\star(t)] = 0$
- Using the optional variation process we found

$$\mathrm{Var}[\widehat{A}(t) - A^\star(t)] = E\left[\int_0^t \frac{J(s)}{Y^2(s)} dN(s)\right]$$

- So an unbiased estimator for $\mathrm{Var}[\widehat{A}(t) - A^\star(t)]$ is

$$\widehat{\sigma}^2(t) = \int_0^t \frac{J(s)}{Y^2(s)} dN(s) = \sum_{j: T_j \le t} \frac{1}{Y^2(T_j^2)}$$

Large sample properties for $\widehat{A}(t)$

- Assume:
  - n individuals
  - each individual has the same hazard rate $\alpha(t)$
  - may have truncation and/or censoring
  - $Y(t)$ is number of individuals at risk just before time t
- Multiplicative intensity process: $\lambda(t) = \alpha(t)Y(t)$
- Assume also
$$\frac{Y(t)}{n} \to y(t) > 0 \text{ when } n \to \infty$$
- Then Rebolledo's theorem gives that
$$\sqrt{n}(\widehat{A}(t) - A^{\star}(t))$$
converges to a Gaussian martingale $U(t)$ with
$$\langle U \rangle(t) = \int_0^t \frac{\alpha(s)}{y(s)} ds$$
- Thus, for large n: $\widehat{A}(t) \approx N(A(t), \sigma^2(t))$

Problem text

- ⋆ Let $N_i = \{N_i(t); t \in [0, \tau]\}$ for $i = 1, \ldots, n$ be n independent nonhomogeneous Poisson processes, all with the same intensity function $\alpha(t)$.
- ⋆ Let $A(t) = \int_0^t \alpha(s)ds$ and let $\{\mathcal{F}_t\}$ contain information about all the N_i processes up to and including time t.
- ⋆ Assume the N_i process is observed only up to time $\tau_i \leq \tau$.
- ⋆ For $t \in [0, \tau]$ define

$$N(t) = \sum_{i=1}^{n} N_i(\min(t, \tau_i)).$$

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a) Show that $\widetilde{N}_i(t) = N_i(\min(t, \tau_i))$ has a multiplicative intensity process.

b) Show that $N(t)$ has a multiplicative intensity process.

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$$N(t) = \sum_{i=1}^{n} N_i(\min(t, \tau_i)).$$

a) Show that $\widetilde{N}_i(t) = N_i(\min(t, \tau_i))$ has a multiplicative intensity process.

b) Show that $N(t)$ has a multiplicative intensity process.

c) Write down an expression for the Nelson-Aalen estimator for $A(t)$ and the associated variance estimator.

Problem text

- Assume $n = 3$, $\tau_1 = 20$, $\tau_2 = 30$ and $\tau_3 = 10$. Assume we have observed event times at 5 12 and 17 for process N_1, 9 and 23 for process N_2, 4 for process N_3.

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d) Calculate the Nelson-Aalen estimator for $A(t)$ by hand, and draw a graph.

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d) Calculate the Nelson-Aalen estimator for $A(t)$ by hand, and draw a graph.

e) Calculate also the estimate for the variance and draw a 95% confidence interval for $A(t)$.

Problem text

- Assume we have observations from two populations
  - assume multiplicative intensity model for each population

$$\lambda_1(t) = \alpha_1(t) Y_1(t) \quad \text{and} \quad \lambda_2(t) = \alpha_2(t) Y_2(t)$$

  - the Nelson-Aalen estimator gives

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- ⋆ Question: How can we evaluate whether it is reasonable to assume

$$\alpha_1(t) = k\alpha_2(t) \;\; \text{for some value } k > 0?$$

  - what type of plot can/should we look at?