Plan for the lecture

- ⋆ Show that $M_n^2 - \langle M \rangle_n$ is a zero-mean martingale
- ⋆ Find the Doob decomposition of a Markov chain
  - – predictable and optional variation processes
- ⋆ Find the Doob decomposition of another chain

Discrete time martingale

- ⋆ Martingale property:

$$\mathsf{E}[M_n|\mathcal{F}_{n-1}] = M_{n-1}, n = 1, 2, \ldots$$

  - equivalently

$$\mathsf{E}[M_n|\mathcal{F}_m] = M_m, n > m$$

- ⋆ Consequences of the martingale property
  - constant mean

$$\mathsf{E}[M_n] = \mathsf{E}[M_0], n = 1, 2, \ldots$$

  - uncorrelated increments

$$\mathsf{Cov}[M_m, M_n - M_m] = 0, n > m$$

Variation processes

- ⋆ Predictable variation process

$$\langle M \rangle_n = \sum_{i=1}^{n} \text{Var}[M_i - M_{i-1} | \mathcal{F}_{i-1}]$$

- ⋆ Optional variation process

$$[M]_n = \sum_{i=1}^{n} (M_i - M_{i-1})^2$$

- ⋆ Consequences av the definitions
  - $M^2 - \langle M \rangle$ is a mean zero martingale
  - $M^2 - [M]$ is a mean zero martingale

Stopping times and transformations

- ⋆ Stopping time T: The event $\{T = t\}$ is only dependent on what happens up (including) to time t
  - stopped process M^T:

$$M_n^T = M_{\min\{n,T\}}$$

- ⋆ H is predictable based on $\{\mathcal{F}_n\}$ if H_n is known based on $\mathcal{F}_{n-1}$.
- ⋆ Transformation of X by H, $Z = H \bullet X$

$$Z_n = H_0 X_0 + H_1(X_1 - X_0) + \ldots + H_n(X_n - X_{n-1})$$

- ⋆ Consequences of the definitions
  - if M mean zero martingale, $H \bullet M$ mean zero martingale
  - $\langle H \bullet M \rangle = H^2 \bullet \langle M \rangle$
  - $[H \bullet M] = H^2 \bullet [M]$

Doob decomposition

- Assume a process X with respect to $\{\mathcal{F}_n\}$, where $X_0 = 0$
- Define M and $X^\star$

$$M_0 = X_0, \Delta M_n = M_n - M_{n-1} = X_n - \mathsf{E}[X_n|\mathcal{F}_{n-1}], n \geq 1$$

$$X_0^\star = 0, X_n^\star = \mathsf{E}[X_n|\mathcal{F}_{n-1}], n \geq 1$$

- Then we have
  - the Doob decomposition

$$X_n = X_n^\star + \Delta M_n$$

  - M is a mean zero martingale
  - $X^\star$ is predictable with respect to $\{\mathcal{F}_n\}$

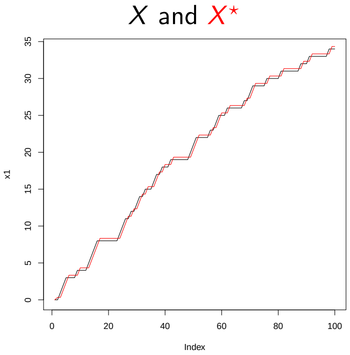
A Markov chain example

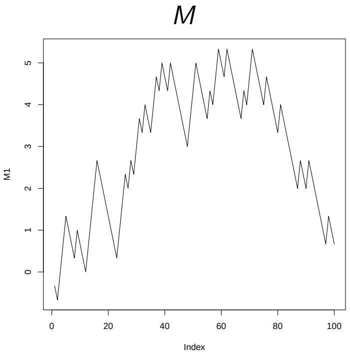
- ⋆ Let $X = (X_0, X_1, X_2, \ldots)$ be a Markov chain with $X_i \in \{0, 1, 2, \ldots\}$, $X_0 = 0$ and

$$P(X_n = x | X_{n-1} = x_{n-1}) = \begin{cases} p & \text{for } x = x_{n-1} + 1 \\ 1 - p & \text{for } x = x_{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

- ⋆ Find the Doob decomposition of X

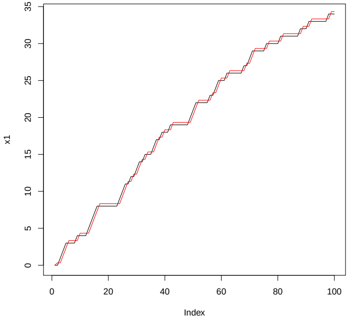
Sample paths



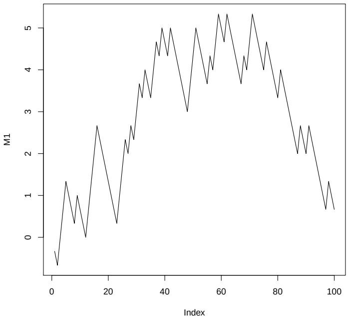


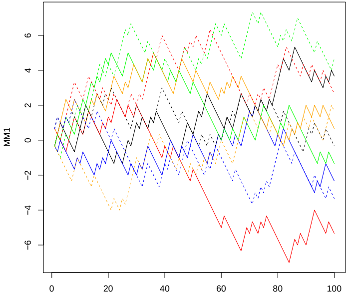
Sample paths

X and $X^{\star}$



M





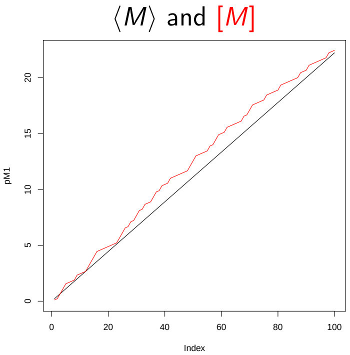
Variational process for M

- Recall:

$$P(X_n = x | X_{n-1} = x_{n-1}) = \begin{cases} p & \text{for } x = x_{n-1} + 1 \\ 1 - p & \text{for } x = x_{n-1} \\ 0 & \text{otherwise.} \end{cases}$$

$$M_n - M_{n-1} = (X_n - X_{n-1}) - p$$

Sample paths of the variational processes



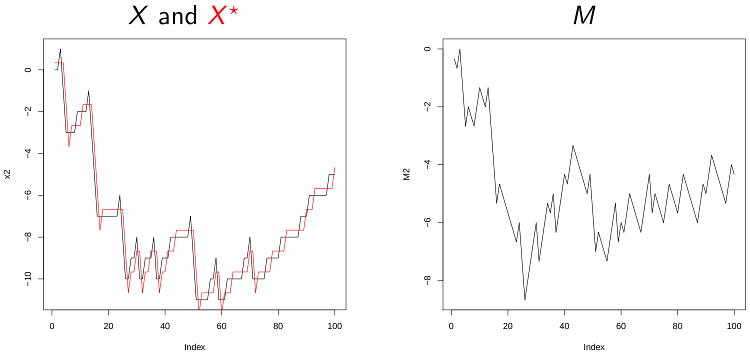
Another stochastic process example

- ⋆ Let $X = (X_0, X_1, X_2, \ldots)$ be a stochastic process with $X_i \in \{0, \pm 1, \pm 2, \ldots\}$, $X_0 = 0$
- ⋆ $U_1, U_2, \ldots$ independent and $P(U_i = 1) = p$, $P(X_i = 0) = 1 - p$
- ⋆ Let $X_1 = X_0 + U_1$, and for $n = 2, 3, \ldots$

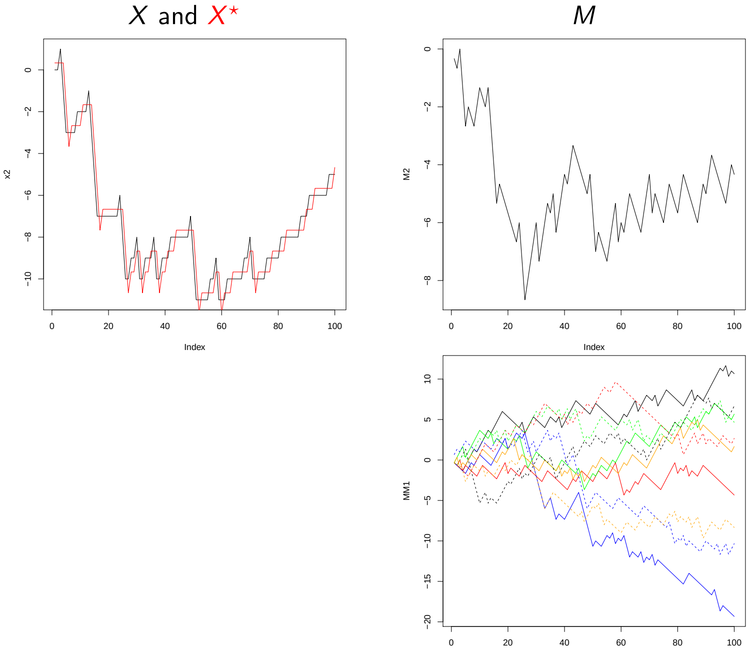
$$X_n = X_{n-1} + U_n(1 - 3U_{n-1})$$

- ⋆ Find the Doob decomposition of X

Sample paths



Sample paths



Variational process for M

- ⋆ Recall: $M_n - M_{n-1} = (U_n - p)(1 - 3U_{n-1})$

Sample paths of the variational processes