

Plan for the lecture

- ★ Consider independent exponentially distributed survival times
 - first without censoring, thereafter with censoring
 - find the Doob-Meyer decomposition
 - find variation processes of the resulting martingale
 - find the variance of the resulting martingale
- ★ Consider another process
 - show that it is a sub-martingale
 - find the Doob-Meyer decomposition
 - sketch a sample path of the compensator and of the martingale

Continuous time martingale

- ★ Martingale property:

$$E[M(t)|\mathcal{F}_s] = M(s) \text{ for } s < t$$

- ★ Consequences of the martingale property

- constant mean

$$E[M(t)] = E[M(0)]$$

- uncorrelated increments

$$\text{Cov}[M(t)-M(s), M(v)-M(u)] = 0 \text{ for } 0 \leq s < t \leq u < v \leq \tau$$

Variation processes

- ★ Predictable variation process

$$\langle M \rangle(t) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \text{Var}[M_k - M_{k-1} | \mathcal{F}_{\frac{(k-1)t}{n}}]$$

- informally: $d\langle M \rangle(t) = \text{Var}[dM(t) | \mathcal{F}_{t-}]$

- ★ Optional variation process

$$[M](t) = \lim_{n \rightarrow \infty} \sum_{k=1}^n (M_k - M_{k-1})^2$$

- ★ Consequences av the definitions

- $M^2 - \langle M \rangle$ is a mean zero martingale
- $M^2 - [M]$ is a mean zero martingale

Covariation processes

- ★ Predictable covariation process

$$\langle M_1, M_2 \rangle(t) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \text{Cov}[\Delta M_{1k}, \Delta M_{2k} | \mathcal{F}_{\frac{(k-1)t}{n}}]$$

- ★ Optional covariation process

$$[M_1, M_2](t) = \lim_{n \rightarrow \infty} \sum_{k=1}^n (\Delta M_{1k})(\Delta M_{2k})$$

- ★ Consequences av the definitions

- $M_1 M_2 - \langle M_1, M_2 \rangle$ is a mean zero martingale
- $M_1 M_2 - [M_1, M_2]$ is a mean zero martingale
- $\langle aM_1, M_2 \rangle = a \langle M_1, M_2 \rangle$ and so on
- $[aM_1, M_2] = a[M_1, M_2]$ and so on
- $\langle M, M \rangle = \langle M \rangle$ and $[M, M] = [M]$

Stochastic integral

★ Stochastic integral

$$\mathbb{I}(t) = \int_0^t H(s) dM(s) = \lim_{n \rightarrow \infty} \sum_{k=1}^n H_k (M_k - M_{k-1})$$

★ Consequences of the definitions

- $\mathbb{I}(t)$ is a mean zero martingale
- $\langle \int H dM \rangle = \int H^2 d\langle M \rangle$
- $[\int H dM] = \int H^2 d[M]$
- $\langle \int H_1 dM_1, \int H_2 dM_2 \rangle = \int H_1 H_2 d\langle M_1, M_2 \rangle$
- $[\int H_1 dM_1, \int H_2 dM_2] = \int H_1 H_2 d[M_1, M_2]$

Doob-Meyer decomposition

- ★ Sub-martingale: X is a sub-martingale if

$$E[X(t)|\mathcal{F}_s] \geq X(s) \quad \text{for } s < t$$

- i.e. $X(t)$ tends to increase

- ★ Doob-Meyer: Assume X is a sub-martingale with respect to $\{\mathcal{F}_t\}$, where $X(0) = 0$. Then there exists a unique decomposition

$$X = X^* + M$$

where

- X^* is non-decreasing and predictable (the compensator for X)
 - M is a zero-mean martingale
- ★ Informally:
 - $dX^*(t) = E[dX(t)|\mathcal{F}_{t-}]$ (predictable from the past)
 - $dM(t) = dX(t) - E[dX(t)|\mathcal{F}_{t-}]$ (innovation/surprise)

The Poisson process

- ★ $N(t)$: number of events in $[0, t]$ in a homogeneous Poisson process with intensity λ
- ★ Doob-Meyer decomposition

$$N(t) = N^*(t) + M(t)$$

where

- $N^*(t) = \lambda t$
- $M(t) = N(t) - \lambda t$

Counting processes

- ★ $N(t)$: A counting process, $N(t)$ adapted to $\{\mathcal{F}_t\}$
 - right continuous
 - jumps of size one
 - constant between jumps

- ★ Recall informal definition of intensity process

$$\lambda(t)dt = P(dN(t) = 1 | \mathcal{F}_{t-}) = E[dN(t) | \mathcal{F}_{t-}]$$

- ★ Precise definition of intensity process by Doob-Meyer

$$N(t) = \Lambda(t) + M(t)$$

- ★ When $\Lambda(t)$ is absolutely continuous

$$\Lambda(t) = \int_0^t \lambda(s)ds$$

- ★ Variation processes

$$\langle M \rangle(t) = \Lambda(t) = \int_0^t \lambda(s)ds \quad \text{and} \quad [M](t) = N(t)$$

Sample paths

