NTNU – Trondheim
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Department of Mathematical Sciences

# Examination paper for **TMA4275 Lifetime Analysis**

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**Examination time (from–to):** 09:00-13:00

**Permitted examination support material:** C: Approved Calculator. One yellow sheet (A4 with stamp) with your own formulae and notes.

**Other information:**
Tables of the $\chi^2$-distribution with 1 degree of freedom, and of the standard normal distribution are enclosed at the end of the exam.

**Language:** English

**Number of pages:** 5

**Number pages enclosed:** 4

**Checked by:**

Date Signature

**Problem 1** *Diesel generator fan*

Let $T$ be the time to failure for a particular type of diesel generator fans. In a sample of 40 fans one has obtained the following data. Here the $Y_i$ are observed times (in hours), while the $\delta_i$ are censoring statuses, where $\delta_i = 1$ if $Y_i$ is a failure time, and $\delta_i = 0$ if it is a right censored time.

| $i$ | $Y_i$ | $\delta_i$ | $i$ | $Y_i$ | $\delta_i$ |
|---|---|---|---|---|---|
| 1 | 450 | 1 | 21 | 4850 | 0 |
| 2 | 1150 | 1 | 22 | 4850 | 0 |
| 3 | 1600 | 0 | 23 | 5000 | 0 |
| 4 | 1660 | 1 | 24 | 5000 | 0 |
| 5 | 1850 | 0 | 25 | 6100 | 0 |
| 6 | 2030 | 0 | 26 | 6100 | 0 |
| 7 | 2070 | 1 | 27 | 6100 | 1 |
| 8 | 2200 | 0 | 28 | 6700 | 0 |
| 9 | 3000 | 0 | 29 | 7450 | 0 |
| 10 | 3000 | 0 | 30 | 7800 | 0 |
| 11 | 3000 | 0 | 31 | 7800 | 0 |
| 12 | 3100 | 1 | 32 | 8100 | 0 |
| 13 | 3200 | 0 | 33 | 8100 | 0 |
| 14 | 3450 | 1 | 34 | 8200 | 0 |
| 15 | 3750 | 0 | 35 | 8500 | 0 |
| 16 | 4150 | 0 | 36 | 8500 | 0 |
| 17 | 4150 | 0 | 37 | 8750 | 0 |
| 18 | 4300 | 0 | 38 | 8750 | 1 |
| 19 | 4300 | 0 | 39 | 8750 | 0 |
| 20 | 4300 | 0 | 40 | 9400 | 0 |

**a)** Let $z(t)$ and $Z(t)$ be, respectively, the *hazard function* and the *cumulative hazard function* for $T$.

Describe briefly how a graph of $Z(t)$ can give us information about the shape of the hazard function $z(t)$.

Calculate and plot the Nelson-Aalen estimator $\hat{Z}_{NA}(t)$ for $Z(t)$ based on the given data.

Does the plot give some indications about the shape of $z(t)$?

Use the plot to calculate an approximate (constant) value for the hazard rate $z(t)$ for the first 3000 hours. How can you use this value to calculate a rough estimate of the expected time to failure (MTTF) for the fan?

**b)** What is the Nelson-Aalen estimate of $Z(1800)$?

Find the standard error for this estimate. Use it to calculate the standard 95% confidence interval for $Z(1800)$.

(*Hint:* The following formula from the lectures can be used:

$$\widehat{Var(\hat{Z}_{NA}(t))} = \sum_{T_{(i)} \leq t} \frac{d_i}{n_i^2} \quad ).$$

One also wants an estimate and a 95% confidence interval for the proportion of fans which survive the first 1800 hours, i.e. $R(1800)$, where $R(t)$ is the reliability function for $T$. Show how you can use the results for $Z(1800)$ calculated above to find both the estimate and the confidence interval for $R(1800)$.

**c)** Let $\mathcal{T}(t)$ be the *Total Time on Test* at time $t$ for the given sample.

The table below shows the value of $\mathcal{T}(t)$ at the 8 failure times.

Explain briefly what the value of $\mathcal{T}(t)$ means, and show in detail how the first three values of $\mathcal{T}(t)$ in the table are calculated.

| $t$ | 450 | 1150 | 1660 | 2070 | 3100 | 3450 | 6100 | 8750 |
|---|---|---|---|---|---|---|---|---|
| $\mathcal{T}(t)$ | 18000 | 45300 | 64620 | 79120 | 111910 | 121460 | 174010 | 200860 |

Draw the TTT-plot based on the table. Does the plot give any indications about the shape of $z(t)$?

Also calculate the test statistic for Barlow-Proschan's test.

Does it lead to rejection of a null hypothesis that $z(t)$ is constant, versus the alternative that it has a monotonic trend? Use significance level 5% for this test.

**d)** Assume in this subproblem that $T$ is exponentially distributed with unknown hazard rate $\lambda$.

The sum of all the 40 observations $Y_i$ is 201 510. Use this information to find the maximum likelihood estimate for $\lambda$. Compare it to the rough estimate of $z(t)$ that you found in subproblem a).

Also, find an estimate, as well as an approximate 95% confidence interval, for $Z(1800)$ under the model assumption in this subproblem.

Compare both the estimate and the confidence interval with the corresponding ones obtained in subproblem b). Give a comment.

**Problem 2** *Repairable system*

We consider a repairable system that operates from time $t = 0$. At failures, the system is repaired and put back into operation immediately.

**a)** Explain briefly what it means that a repair is, respectively, *minimal* and *perfect*.

Which of these two types of repair is modeled by NHPP? Give a short explanation why an NHPP is an appropriate model in this case.

What kind of model is used for the other type of repair? Give a short explanation.

Let $N(t)$ denote the number of failures in the time interval $(0, t]$. In the following we assume that $\{N(t) : t > 0\}$ is a non-homogeneous Poisson process (NHPP) with intensity function (ROCOF) $w(t; \theta)$ and cumulative intensity function (CROCOF) $W(t; \theta) = \int_0^t w(u; \theta)du$. Here $\theta$ is a parameter, possibly a vector.

In order to do statistical inference about $\theta$, one observes the system in the time interval $[0, \tau]$, for some given $\tau > 0$. The data consist of number of failures $D_i$ in time intervals $(h_{i-1}, h_i]$, for $i = 1, 2, \ldots, r$, where

$$h_0 = 0 < h_1 < h_2 < \cdots < h_r = \tau.$$

**b)** Explain why we can conclude that $D_1, \ldots, D_r$ are independent and Poisson-distributed, with

$$E(D_i) = W(h_i; \theta) - W(h_{i-1}; \theta) \text{ for } i = 1, 2 \ldots, r.$$

Use this to show that the likelihood function for $\theta$, when the observations of $D_1, \ldots, D_r$ are $d_1, \ldots, d_r$, can be written

$$L(\theta) = \left\{ \prod_{i=1}^{r} \frac{[W(h_i; \theta) - W(h_{i-1}; \theta)]^{d_i}}{d_i!} \right\} e^{-W(\tau;\theta)}. \tag{1}$$

For a machine one has recorded number of failures per week for $r = 10$ weeks. Let the weeks be numbered $1, 2, \ldots, 10$, and let $d_i$ be the number of failures observed in week number $i$, $i = 1, \ldots, 10$.

It is assumed that the failures occur as an NHPP with cumulative intensity function

$$W(t;\lambda,\beta) = \lambda t^{\beta} \text{ for } t > 0,$$

where the time unit is *weeks*, and $\lambda > 0, \beta > 0$ are unknown parameters.

The observed data are given by the following table:

| Week no. $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Number of failures $d_i$ | 0 | 3 | 4 | 5 | 3 | 1 | 3 | 5 | 7 | 4 |

**c)** Use the formula (1) to show that the log-likelihood of the given data can be written as

$$\ell(\lambda,\beta) = (\ln\lambda)\sum_{i=1}^{10} d_i + \sum_{i=1}^{10} d_i \ln\left(i^{\beta} - (i-1)^{\beta}\right) - \sum_{i=1}^{10} \ln(d_i!) - \lambda \cdot 10^{\beta}.$$

(*Hint:* Let $h_i = i$ for $i = 0, 1, \ldots, 10$).

Show that if $\beta$ is known, then the maximum likelihood estimator for $\lambda$ is

$$\hat{\lambda}(\beta) = \frac{\sum_{i=1}^{10} d_i}{10^{\beta}}.$$

Compute the value of $\hat{\lambda}(\beta)$ with the given data if it is known that $\beta = 1.3$.

With these values of $\beta$ and $\lambda$, what are the probabilities of no failures in, respectively, the first week and in the 10th week?

**d)** Calculate the profile log likelihood function $\tilde{\ell}(\beta)$ of $\beta$.

A graph of $\tilde{\ell}(\beta)$ with the given data is given on the next page.

You may use that the maximum value of $\tilde{\ell}(\beta)$ is $-18.60$.

Use the graph (in a rough manner)

1. to find the maximum likelihood estimate of $\beta$,
2. to find an approximate 95% confidence interval of $\beta$,
3. to test, with significance level 5%, the null hypothesis of no time trend in the occurrence of failures of the machine.

Find also the maximum likelihood estimate of $\lambda$.

*(In your exam paper you may illustrate all this by a rough sketch of the graph).*

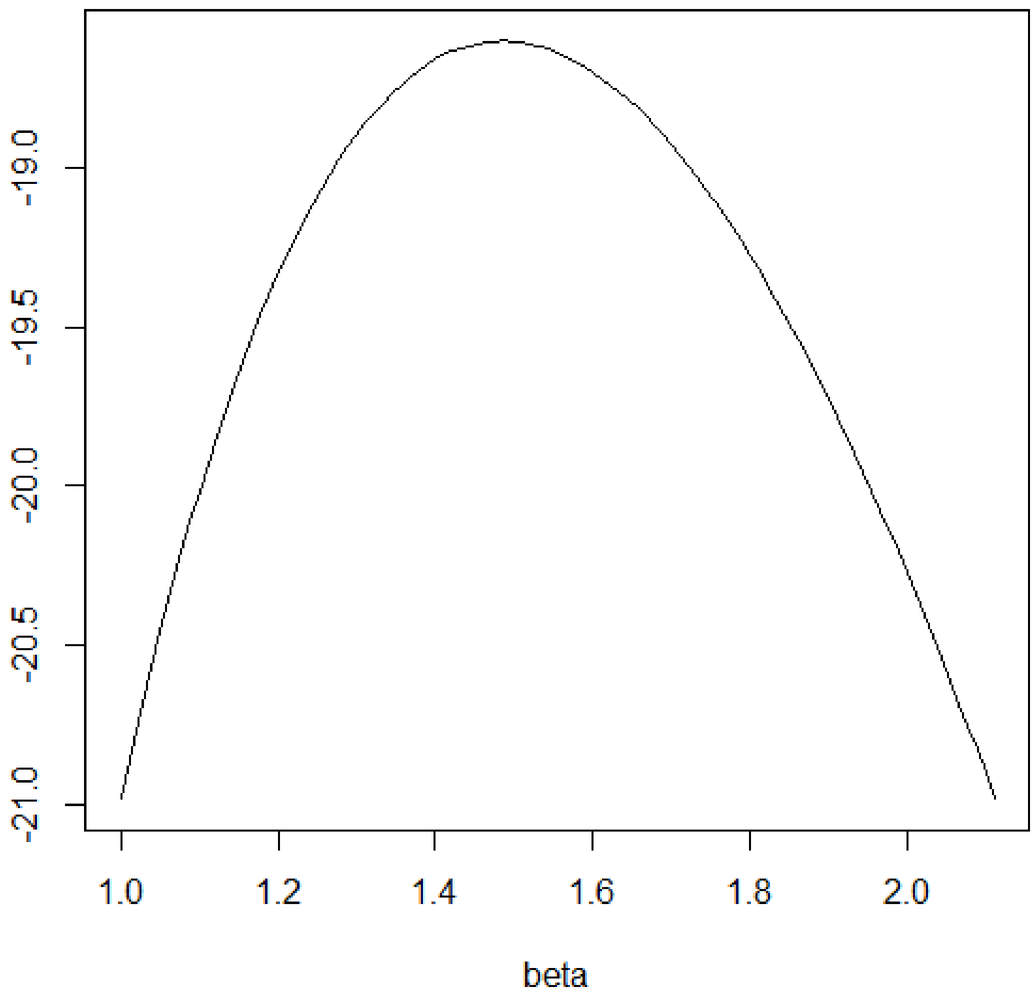
-19.0
-19.5
-20.0
-20.5
-21.0
1.0
1.2
1.4
1.6
1.8
2.0
beta

**Table of cumulative probabilities of the $\chi^2$-distribution with 1 degree of freedom.**

| $x$ | $P(X \leq x)$ | $x$ | $P(X \leq x)$ |
|---|---|---|---|
| 0.01 | 0.079656 | 2.7 | 0.899652 |
| 0.02 | 0.112463 | 2.8 | 0.905736 |
| 0.03 | 0.137510 | 2.9 | 0.911420 |
| 0.04 | 0.158519 | 3.0 | 0.916735 |
| 0.05 | 0.176937 | 3.1 | 0.921708 |
| 0.06 | 0.193504 | 3.2 | 0.926362 |
| 0.07 | 0.208663 | 3.3 | 0.930720 |
| 0.08 | 0.222703 | 3.4 | 0.934804 |
| 0.09 | 0.235823 | 3.5 | 0.938631 |
| 0.10 | 0.248170 | 3.6 | 0.942220 |
| 0.20 | 0.345279 | 3.7 | 0.945588 |
| 0.30 | 0.416118 | 3.8 | 0.948747 |
| 0.40 | 0.472911 | 3.9 | 0.951714 |
| 0.50 | 0.520500 | 4.0 | 0.954500 |
| 0.60 | 0.561422 | 4.1 | 0.957117 |
| 0.70 | 0.597216 | 4.2 | 0.959576 |
| 0.80 | 0.628907 | 4.3 | 0.961888 |
| 0.90 | 0.657218 | 4.4 | 0.964061 |
| 1.00 | 0.682689 | 4.5 | 0.966105 |
| 1.10 | 0.705734 | 4.6 | 0.968028 |
| 1.20 | 0.726678 | 4.7 | 0.969837 |
| 1.30 | 0.745787 | 4.8 | 0.971540 |
| 1.40 | 0.763276 | 4.9 | 0.973143 |
| 1.50 | 0.779329 | 5.0 | 0.974653 |
| 1.60 | 0.794097 | 5.1 | 0.976074 |
| 1.70 | 0.807712 | 5.2 | 0.977413 |
| 1.80 | 0.820288 | 5.3 | 0.978675 |
| 1.90 | 0.831922 | 5.4 | 0.979863 |
| 2.00 | 0.842701 | 5.5 | 0.980984 |
| 2.10 | 0.852701 | 5.6 | 0.982040 |
| 2.20 | 0.861989 | 5.7 | 0.983035 |
| 2.30 | 0.870626 | 5.8 | 0.983974 |
| 2.40 | 0.878665 | 5.9 | 0.984859 |
| 2.50 | 0.886154 | 6.0 | 0.985694 |
| 2.60 | 0.893136 | 6.1 | 0.986482 |

## Standard normalfordeling

$$\Phi(z) = P(Z \leq z)$$

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| -3.7 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 |
| -3.6 | .0002 | .0002 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 | .0001 |
| -3.5 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 | .0002 |
| -3.4 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0003 | .0002 |
| -3.3 | .0005 | .0005 | .0005 | .0004 | .0004 | .0004 | .0004 | .0004 | .0004 | .0003 |
| -3.2 | .0007 | .0007 | .0006 | .0006 | .0006 | .0006 | .0006 | .0005 | .0005 | .0005 |
| -3.1 | .0010 | .0009 | .0009 | .0009 | .0008 | .0008 | .0008 | .0008 | .0007 | .0007 |
| -3.0 | .0013 | .0013 | .0013 | .0012 | .0012 | .0011 | .0011 | .0011 | .0010 | .0010 |
| -2.9 | .0019 | .0018 | .0018 | .0017 | .0016 | .0016 | .0015 | .0015 | .0014 | .0014 |
| -2.8 | .0026 | .0025 | .0024 | .0023 | .0023 | .0022 | .0021 | .0021 | .0020 | .0019 |
| -2.7 | .0035 | .0034 | .0033 | .0032 | .0031 | .0030 | .0029 | .0028 | .0027 | .0026 |
| -2.6 | .0047 | .0045 | .0044 | .0043 | .0041 | .0040 | .0039 | .0038 | .0037 | .0036 |
| -2.5 | .0062 | .0060 | .0059 | .0057 | .0055 | .0054 | .0052 | .0051 | .0049 | .0048 |
| -2.4 | .0082 | .0080 | .0078 | .0075 | .0073 | .0071 | .0069 | .0068 | .0066 | .0064 |
| -2.3 | .0107 | .0104 | .0102 | .0099 | .0096 | .0094 | .0091 | .0089 | .0087 | .0084 |
| -2.2 | .0139 | .0136 | .0132 | .0129 | .0125 | .0122 | .0119 | .0116 | .0113 | .0110 |
| -2.1 | .0179 | .0174 | .0170 | .0166 | .0162 | .0158 | .0154 | .0150 | .0146 | .0143 |
| -2.0 | .0228 | .0222 | .0217 | .0212 | .0207 | .0202 | .0197 | .0192 | .0188 | .0183 |
| -1.9 | .0287 | .0281 | .0274 | .0268 | .0262 | .0256 | .0250 | .0244 | .0239 | .0233 |
| -1.8 | .0359 | .0351 | .0344 | .0336 | .0329 | .0322 | .0314 | .0307 | .0301 | .0294 |
| -1.7 | .0446 | .0436 | .0427 | .0418 | .0409 | .0401 | .0392 | .0384 | .0375 | .0367 |
| -1.6 | .0548 | .0537 | .0526 | .0516 | .0505 | .0495 | .0485 | .0475 | .0465 | .0455 |
| -1.5 | .0668 | .0655 | .0643 | .0630 | .0618 | .0606 | .0594 | .0582 | .0571 | .0559 |
| -1.4 | .0808 | .0793 | .0778 | .0764 | .0749 | .0735 | .0721 | .0708 | .0694 | .0681 |
| -1.3 | .0968 | .0951 | .0934 | .0918 | .0901 | .0885 | .0869 | .0853 | .0838 | .0823 |
| -1.2 | .1151 | .1131 | .1112 | .1093 | .1075 | .1056 | .1038 | .1020 | .1003 | .0985 |
| -1.1 | .1357 | .1335 | .1314 | .1292 | .1271 | .1251 | .1230 | .1210 | .1190 | .1170 |
| -1.0 | .1587 | .1562 | .1539 | .1515 | .1492 | .1469 | .1446 | .1423 | .1401 | .1379 |
| -.9 | .1841 | .1814 | .1788 | .1762 | .1736 | .1711 | .1685 | .1660 | .1635 | .1611 |
| -.8 | .2119 | .2090 | .2061 | .2033 | .2005 | .1977 | .1949 | .1922 | .1894 | .1867 |
| -.7 | .2420 | .2389 | .2358 | .2327 | .2296 | .2266 | .2236 | .2206 | .2177 | .2148 |
| -.6 | .2743 | .2709 | .2676 | .2643 | .2611 | .2578 | .2546 | .2514 | .2483 | .2451 |
| -.5 | .3085 | .3050 | .3015 | .2981 | .2946 | .2912 | .2877 | .2843 | .2810 | .2776 |
| -.4 | .3446 | .3409 | .3372 | .3336 | .3300 | .3264 | .3228 | .3192 | .3156 | .3121 |
| -.3 | .3821 | .3783 | .3745 | .3707 | .3669 | .3632 | .3594 | .3557 | .3520 | .3483 |
| -.2 | .4207 | .4168 | .4129 | .4090 | .4052 | .4013 | .3974 | .3936 | .3897 | .3859 |
| -.1 | .4602 | .4562 | .4522 | .4483 | .4443 | .4404 | .4364 | .4325 | .4286 | .4247 |
| -.0 | .5000 | .4960 | .4920 | .4880 | .4840 | .4801 | .4761 | .4721 | .4681 | .4641 |

## Standard normalfordeling

$$\Phi(z) = P(Z \leq z)$$

| $z$ | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| .0 | .5000 | .5040 | .5080 | .5120 | .5160 | .5199 | .5239 | .5279 | .5319 | .5359 |
| .1 | .5398 | .5438 | .5478 | .5517 | .5557 | .5596 | .5636 | .5675 | .5714 | .5753 |
| .2 | .5793 | .5832 | .5871 | .5910 | .5948 | .5987 | .6026 | .6064 | .6103 | .6141 |
| .3 | .6179 | .6217 | .6255 | .6293 | .6331 | .6368 | .6406 | .6443 | .6480 | .6517 |
| .4 | .6554 | .6591 | .6628 | .6664 | .6700 | .6736 | .6772 | .6808 | .6844 | .6879 |
| .5 | .6915 | .6950 | .6985 | .7019 | .7054 | .7088 | .7123 | .7157 | .7190 | .7224 |
| .6 | .7257 | .7291 | .7324 | .7357 | .7389 | .7422 | .7454 | .7486 | .7517 | .7549 |
| .7 | .7580 | .7611 | .7642 | .7673 | .7704 | .7734 | .7764 | .7794 | .7823 | .7852 |
| .8 | .7881 | .7910 | .7939 | .7967 | .7995 | .8023 | .8051 | .8078 | .8106 | .8133 |
| .9 | .8159 | .8186 | .8212 | .8238 | .8264 | .8289 | .8315 | .8340 | .8365 | .8389 |
| 1.0 | .8413 | .8438 | .8461 | .8485 | .8508 | .8531 | .8554 | .8577 | .8599 | .8621 |
| 1.1 | .8643 | .8665 | .8686 | .8708 | .8729 | .8749 | .8770 | .8790 | .8810 | .8830 |
| 1.2 | .8849 | .8869 | .8888 | .8907 | .8925 | .8944 | .8962 | .8980 | .8997 | .9015 |
| 1.3 | .9032 | .9049 | .9066 | .9082 | .9099 | .9115 | .9131 | .9147 | .9162 | .9177 |
| 1.4 | .9192 | .9207 | .9222 | .9236 | .9251 | .9265 | .9279 | .9292 | .9306 | .9319 |
| 1.5 | .9332 | .9345 | .9357 | .9370 | .9382 | .9394 | .9406 | .9418 | .9429 | .9441 |
| 1.6 | .9452 | .9463 | .9474 | .9484 | .9495 | .9505 | .9515 | .9525 | .9535 | .9545 |
| 1.7 | .9554 | .9564 | .9573 | .9582 | .9591 | .9599 | .9608 | .9616 | .9625 | .9633 |
| 1.8 | .9641 | .9649 | .9656 | .9664 | .9671 | .9678 | .9686 | .9693 | .9699 | .9706 |
| 1.9 | .9713 | .9719 | .9726 | .9732 | .9738 | .9744 | .9750 | .9756 | .9761 | .9767 |
| 2.0 | .9772 | .9778 | .9783 | .9788 | .9793 | .9798 | .9803 | .9808 | .9812 | .9817 |
| 2.1 | .9821 | .9826 | .9830 | .9834 | .9838 | .9842 | .9846 | .9850 | .9854 | .9857 |
| 2.2 | .9861 | .9864 | .9868 | .9871 | .9875 | .9878 | .9881 | .9884 | .9887 | .9890 |
| 2.3 | .9893 | .9896 | .9898 | .9901 | .9904 | .9906 | .9909 | .9911 | .9913 | .9916 |
| 2.4 | .9918 | .9920 | .9922 | .9925 | .9927 | .9929 | .9931 | .9932 | .9934 | .9936 |
| 2.5 | .9938 | .9940 | .9941 | .9943 | .9945 | .9946 | .9948 | .9949 | .9951 | .9952 |
| 2.6 | .9953 | .9955 | .9956 | .9957 | .9959 | .9960 | .9961 | .9962 | .9963 | .9964 |
| 2.7 | .9965 | .9966 | .9967 | .9968 | .9969 | .9970 | .9971 | .9972 | .9973 | .9974 |
| 2.8 | .9974 | .9975 | .9976 | .9977 | .9977 | .9978 | .9979 | .9979 | .9980 | .9981 |
| 2.9 | .9981 | .9982 | .9982 | .9983 | .9984 | .9984 | .9985 | .9985 | .9986 | .9986 |
| 3.0 | .9987 | .9987 | .9987 | .9988 | .9988 | .9989 | .9989 | .9989 | .9990 | .9990 |
| 3.1 | .9990 | .9991 | .9991 | .9991 | .9992 | .9992 | .9992 | .9992 | .9993 | .9993 |
| 3.2 | .9993 | .9993 | .9994 | .9994 | .9994 | .9994 | .9994 | .9995 | .9995 | .9995 |
| 3.3 | .9995 | .9995 | .9995 | .9996 | .9996 | .9996 | .9996 | .9996 | .9996 | .9997 |
| 3.4 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9997 | .9998 |
| 3.5 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 | .9998 |
| 3.6 | .9998 | .9998 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |
| 3.7 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 | .9999 |

## Kritiske verdier i standard normalfordelingen

$$P(Z > z_\alpha) = \alpha$$

| $\alpha$ | $z_\alpha$ |
|---|---|
| .2 | 0.842 |
| .15 | 1.036 |
| .1 | 1.282 |
| .075 | 1.440 |
| .05 | 1.645 |
| .04 | 1.751 |
| .03 | 1.881 |
| .025 | 1.960 |
| .02 | 2.054 |
| .01 | 2.326 |
| .005 | 2.576 |
| .001 | 3.090 |
| .0005 | 3.291 |
| .0001 | 3.719 |
| .00005 | 3.891 |
| .00001 | 4.265 |
| .000005 | 4.417 |
| .000001 | 4.753 |