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Examination paper for **TMA4275 Lifetime Analysis**

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Examination date: June 7th 2022 – solution sketch
Examination time (from–to): 09.00-13.00
Permitted examination support material: A

Other information:

Language: English
Number of pages: 10
Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

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Problem 1

The Nelson-Aalen estimator is

$$\widehat{A}(t) = \sum_{j:T_j \leq t} \frac{1}{Y(T_j)} \tag{1}$$

The evaluation of the estimator is shown in Table 1.

Table 1: Evaluation of the Nelson-Aalen estimator.

Time	$Y(t)$	$\frac{1}{Y(t)}$	$\widehat{A}(t)$
1.11	10	0.1	0.1
1.35	8	0.125	0.1+0.125=0.225
2.16	5	0.2	0.225+0.2=0.425
2.22	4	0.25	0.425+0.25=0.675
2.40	2	0.5	0.675+0.5 = 1.175

The estimator for the variance of the Nelson-Aalen estimator is

$$\widehat{\sigma}^2(t) = \sum_{j:T_j \leq t} \frac{1}{Y(T_j)^2} \tag{2}$$

and the approximate 95% confidence interval based on the log transformation at time t is

$$\left[\widehat{A}(t)\exp\left\{-z_{0.025}\frac{\sqrt{\widehat{\sigma}^2(t)}}{\widehat{A}(t)}\right\}, \widehat{A}(t)\exp\left\{z_{0.025}\frac{\sqrt{\widehat{\sigma}^2(t)}}{\widehat{A}(t)}\right\}\right]$$

where $z_{0.025} = 1.96$.

Possible R code for solving the remaining part of the problem and the associated R output are given in the following.

```
> Y = c(10,8,5,4,2)
> Ahat = c(0.1,0.225,0.425,0.675,1.175)
>
> Sigma2hat = cumsum(1/(Y^2))
> Sigma2hat
[1] 0.010000 0.025625 0.065625 0.128125 0.378125
>
> LowerLimit = Ahat * exp(-1.96*sqrt(Sigma2hat)/Ahat)
```

```
> UpperLimit = Ahat * exp(+1.96*sqrt(Sigma2hat)/Ahat)
> LowerLimit
[1] 0.01408584 0.05579266 0.13040905 0.23873280 0.42127537
> UpperLimit
[1] 0.7099327 0.9073775 1.3850650 1.9085144 3.2772507
>
> x = c(0,1.11,1.11,1.35,1.35,2.16,2.16,2.22,2.22,2.40,2.40,3.0)
> A = Ahat
> yA = c(0,0,A[1],A[1],A[2],A[2],A[3],A[3],A[4],A[4],A[5],A[5])
> L = LowerLimit
> yLower = c(0,0,L[1],L[1],L[2],L[2],L[3],L[3],L[4],L[4],L[5],L[5])
> U = UpperLimit
> yUpper = c(0,0,U[1],U[1],U[2],U[2],U[3],U[3],U[4],U[4],U[5],U[5])
>
> pdf("NA.pdf")
> plot(x,yA,ylim=c(0,max(UpperLimit)*1.05),cex.axis=1.5,type="l",
  xlab="t",ylab="A(t)")
> lines(x,yLower,col="red")
> lines(x,yUpper,col="red")
> graphics.off()
```

The resulting plot is shown in Figure 1.

Problem 2

We start by finding the integrated hazard rate,

$$\begin{aligned} A(t) &= \int_0^t \alpha(u)du = \int_0^t \lambda e^{\beta u} du \\ &= \left[\frac{\lambda}{\beta} e^{\beta u}\right]_0^t = \frac{\lambda}{\beta} e^{\beta t} - \frac{\lambda}{\beta} e^0 \\ &= \frac{\lambda}{\beta}\left(e^{\beta t} - 1\right). \end{aligned}$$

We can then find the survival function $S(t)$,

$$\underline{\underline{S(t)}} = e^{-A(t)} = \underline{\underline{\exp\left\{-\frac{\lambda}{\beta}\left(e^{\beta t} - 1\right)\right\}}}.$$

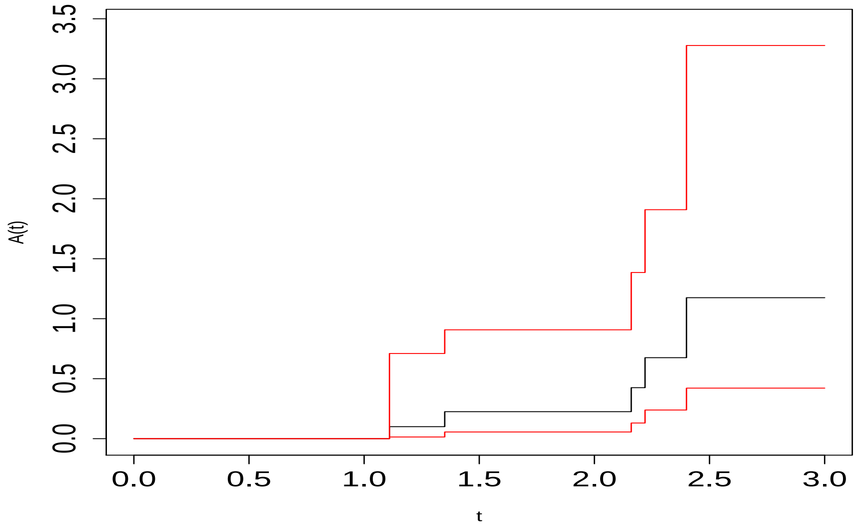

Figure 1: Estimated integrated hazard rate and associated 95% confidence interval for $t \in [0, 3]$.

The density function is then given by differentiation. Using the chain rule we get

$$\begin{aligned} f(t) = -S'(t) &= -\exp\left\{-\frac{\lambda}{\beta}\left(e^{\beta t}-1\right)\right\} \cdot \left(-\frac{\lambda}{\beta}\cdot \beta e^{\beta t}\right) \\ &= \underline{\underline{\lambda \exp\left\{\beta t - \frac{\lambda}{\beta}\left(e^{\beta t}-1\right)\right\}}}. \end{aligned}$$

Problem 3

The estimated relative risk function is

$$\underline{\underline{r(x, \widehat{\beta}) = \exp\{0.066748 \cdot \text{age} + 0.236694 \cdot \mathbb{I}(\text{sex=male}) - 0.084901 \cdot \text{mspike}\}}}.$$

The ratio of the hazard rate for a female of age 50 years over the hazard rate of a female of age 51 years, when the mspike value is the same, becomes

$$\frac{e^{\beta_{\text{age}}\cdot 50 + \beta_{\text{sex}}\cdot 0 + \beta_{\text{mspike}}\cdot \text{mspike}}}{e^{\beta_{\text{age}}\cdot 51 + \beta_{\text{sex}}\cdot 0 + \beta_{\text{mspike}}\cdot \text{mspike}}} = e^{-\beta_{\text{age}}}.$$

From the R output we find a 95% confidence interval for β_{age} to be

$[0.066748 - 1.96 \cdot 0.007051, 0.066748 + 1.96 \cdot 0.007051] = [0.05292804, 0.08056796]$.

Since $e^{-\beta_{\text{age}}}$ is strictly decreasing as a function of β_{age} the corresponding 95% confidence interval for $e^{-\beta_{\text{age}}}$ becomes

$$\left[e^{-0.08056796}, e^{-0.05292804}\right] = \underline{\underline{[0.9225922, 0.9484483]}}.$$

The relative risk function it is asked for is

$$r = \exp\left\{\beta_{\text{age}} \cdot 35 + \beta_{\text{sex}} \cdot 1 + \beta_{\text{mspike}} \cdot 1.5\right\}.$$

The estimate for r is

$$\begin{aligned}\widehat{r} &= \exp\left\{\widehat{\beta}_{\text{age}} \cdot 35 + \widehat{\beta}_{\text{sex}} \cdot 1 + \widehat{\beta}_{\text{mspike}} \cdot 1.5\right\} \\ &= \exp\{0.066748 \cdot 35 + 0.236694 \cdot 1 - 0.084901 \cdot 1.5\} \\ &= \exp\{2.445523\}.\end{aligned}$$

Using that the vector of estimators, $\widehat{\beta}$, are approximately multivariate normal with a covariance matrix as given in the R output we get that $\ln(\widehat{r})$ is also approximately normal with mean equal to $\ln(r)$ and variance

$$\begin{aligned}\text{Var}[\ln(\widehat{r})] &= \begin{bmatrix} 35 \\ 1 \\ 1.5 \end{bmatrix}^T \cdot \begin{bmatrix} 4.972337e-05 & -7.378061e-05 & 4.857322e-05 \\ -7.378061e-05 & 1.871872e-02 & 1.957005e-04 \\ 4.857322e-05 & 1.957005e-04 & 2.922280e-02 \end{bmatrix} \cdot \begin{bmatrix} 35 \\ 1 \\ 1.5 \end{bmatrix} \\ &= 0.1459038.\end{aligned}$$

A 95% confidence interval for $\ln(r)$ is thereby

$$\begin{aligned}&[\ln(\widehat{r}) - 1.96\sqrt{0.1459038}, \ln(\widehat{r}) - 1.96\sqrt{0.1459038}] \\ &= [2.445523 - 1.96\sqrt{0.1459038}, 2.445523 + 1.96\sqrt{0.1459038} \\ &= [1.696855, 3.194191].\end{aligned}$$

A 95% confidence interval for $r = e^{\ln(r)}$ is then

$$[\exp\{1.696855\}, \exp\{3.194191\}] = \underline{\underline{[5.456759, 24.39043]}}.$$

Problem 4

To find an expression for the partial likelihood we start with the general expression for partial likelihood for a relative risk regression model, given in (4.7) in ABG,

$$L(\beta) = \prod_{T_j} \frac{r(\beta, x_{i_j}(T_j))}{\sum_{\ell \in \mathcal{R}_j} r(\beta, x_\ell(T_j))}.$$

Since our covariates are time invariant we can simplify this expression by inserting $x_{i_j}(T_j) = x_{i_j}$ and $x_\ell(T_j) = x_\ell$. Since we have only one covariate we have that β is scalar, so $r(\beta, x) = e^{\beta x}$ we then get

$$L(\beta) = \prod_{T_j} \frac{e^{\beta x_{i_j}}}{\sum_{\ell \in \mathcal{R}_j} e^{\beta x_\ell}}.$$

Since all the units are at risk at all times we get that $\mathcal{R}_j = \{1, 2, \ldots, n\}$ so

$$L(\beta) = \prod_{T_j} \frac{e^{\beta x_{i_j}}}{\sum_{\ell=1}^{n} e^{\beta x_\ell}}.$$

To simplify the denominator in this expression we can define

$$\mathcal{B}_1 = \{i : x_i = 1\}$$

to be the set of units that have the covariate equal to one, and correspondingly define

$$\mathcal{B}_0 = \{i : x_i = 0\} = \{1, 2, \ldots, n\} \setminus \mathcal{B}_1$$

to be the set of units that have the covariate equal to zero. Moreover, let $z_1 = |\mathcal{B}_1|$ be the number of units that has the covariate equal to one, and let $z_0 = |\mathcal{B}_0| = n - z_1$. Since the covariate value is binary, the denominator can then be written as

$$\sum_{\ell=1}^{n} e^{\beta x_\ell} = \sum_{\ell : x_\ell = 0} e^{\beta \cdot 0} + \sum_{\ell : x_\ell = 1} e^{\beta \cdot 1} = \sum_{\ell : x_\ell = 0} 1 + \sum_{\ell : x_\ell = 1} e^{\beta} = z_0 + z_1 e^{\beta}.$$

The partial likelihood function can thereby be expressed as

$$L(\beta) = \prod_{T_j} \frac{e^{\beta x_{i_j}}}{z_0 + z_1 e^{\beta}}.$$

Taking the logarithm of this expression we get the log-partial likelihood function,

$$\ell(\beta) = \sum_{T_j} \left(\beta x_{i_j} - \ln(z_0 + z_1 e^{\beta}) \right).$$

Letting K_0 and K_1 denote the number of failures observed up to time τ on units with covariate value equal to zero and one, respectively, and letting $K = K_0 + K_1$ denote the total number of failures observed up to time τ, the expression for $\ell(\beta)$ can be written as

$$\begin{aligned}
\ell(\beta) &= \sum_{T_j} \beta x_{i_j} - \sum_{T_j} \ln(z_0 + z_1 e^{\beta}) \\
&= \sum_{T_j : x_{i_j} = 0} \beta \cdot 0 + \sum_{T_j : x_{i_j} = 1} \beta \cdot 1 - K \ln(z_0 + z_1 e^{\beta}) \\
&= K_0 \cdot 0 + K_1 \cdot \beta - K \ln(z_0 + z_1 e^{\beta}) \\
&= \beta K_1 - K \ln(z_0 + z_1 e^{\beta}).
\end{aligned}$$

Using the chain rule to evaluate the derivative of $\ell(\beta)$ we get

$$\ell'(\beta) = K_1 - K \frac{1}{z_0 + z_1 e^{\beta}} \cdot z_1 e^{\beta}$$

Setting $\ell'(\beta)$ to find the maximum likelihood estimator for β we get

$$\begin{aligned}
K_1 &= \frac{K z_1 e^{\beta}}{z_0 + z_1 e^{\beta}} \\
K_1(z_0 + z_1 e^{\beta}) &= K z_1 e^{\beta} \\
K_1 z_0 &= (K - K_1) z_1 e^{\beta} \\
e^{\beta} &= \frac{K_1 z_0}{(K - K_1) z_1} = \frac{K_1 z_0}{K_0 z_1} \\
\beta &= \ln\left(\frac{K_1 z_0}{K_0 z_1}\right).
\end{aligned}$$

So the maximum partial likelihood for β is

$$\underline{\underline{\widehat{\beta} = \ln\left(\frac{K_1 z_0}{K_0 z_1}\right)}}.$$

Problem 5

a) The Laplace transform of a stochastic variable Z is defined as

$$\mathscr{L}(c) = \mathrm{E}[e^{-cZ}].$$

For a Z that are exponentially distributed with mean $1/\lambda$ we then get

$$\begin{aligned}
\mathscr{L}(c) &= \int_0^\infty e^{-cz}\lambda e^{-\lambda z} dz \\
&= \lambda \int_0^\infty e^{-(\lambda+c)z} dz \\
&= \lambda \left[-\frac{1}{\lambda+c} e^{-(\lambda+c)z} \right]_0^\infty \\
&= \lambda \cdot \left(-\frac{1}{\lambda+c} \cdot 0 - \left(-\frac{1}{\lambda+c} \cdot e^0 \right) \right) \\
&= \underline{\underline{\frac{\lambda}{\lambda+c}}}
\end{aligned}$$

To show the given formula for $\mathscr{L}^{(r)}(c)$ we first check that it is correct for $r = 0$,

$$\mathscr{L}^{(0)}(c) = (-1)^0 \frac{\lambda \cdot 0!}{(\lambda+c)^{0+1}} = \frac{\lambda}{\lambda+c},$$

which is equal to $\mathscr{L}(c)$ as it should. Then we do the induction step, i.e. we assume the given formula to be correct for $r = s$ and use this to check the formula for $r = s+1$. As we now have assumed the formula to be correct for $r = s$ we have that

$$\begin{aligned}
\mathscr{L}^{(s)}(c) &= (-1)^s \frac{\lambda \cdot s!}{(\lambda+c)^{s+1}} \\
&= (-1)^s \lambda (s!) (\lambda+c)^{-(s+1)}.
\end{aligned}$$

Taking the derivative of this expression with respect to c we get

$$\begin{aligned}
\mathscr{L}^{(s+1)}(c) &= (-1)^s \lambda (s!)(-(s+1))(\lambda+c)^{-(s+1)-1} \\
&= (-1)^{s+1} \lambda ((s+1)!)(\lambda+c)^{-((s+1)+1)} \\
&= (-1)^{s+1} \frac{\lambda \cdot (s+1)!}{(\lambda+c)^{(s+1)+1}}.
\end{aligned}$$

We see that this last expression is identical to what we get by inserting $r = s+1$ in the given expression for $\mathscr{L}^{(r)}(c)$, and we have thereby shown that the given formula is correct for $r = 0, 1, 2, \ldots$.

b) The situation given in the problem text is identical to the situation discussed in Section 7.2.2 in ABG, so the log-likelihood is given by (7.3) in ABG,

$$\ell(\lambda, k) = \sum_{i=1}^m \left[\sum_{j=1}^{n_i} D_{ij} \ln(\alpha(\widetilde{T}_{ij})) + \ln\left((-1)^{D_{i\bullet}} \mathscr{L}^{(D_{i\bullet})}(V_i) \right) \right],$$

where

$$V_i = \sum_{j=1}^{n_i} A(\widetilde{T}_{ij}).$$

From the given formula for $\alpha(t|Z)$ we have that $\alpha(t) = t^{k-1}$, which gives

$$A(t) = \int_0^t \alpha(u)du = \int_0^t u^{k-1}du = \left[\frac{u^k}{k}\right]_0^t = \frac{t^k}{k},$$

which in turn gives

$$V_i = \sum_{j=1}^{n_i} \frac{\widetilde{T}^k}{k} = \frac{1}{k}\sum_{j=1}^{n_i} \widetilde{T}_{ij}^k.$$

Inserting the expressions we have for $\alpha(t)$, V_i and $\mathscr{L}^{(r)}(c)$ into the expression for $\ell(\lambda, k)$ we get

$$\begin{aligned}
\ell(\lambda, k) &= \sum_{i=1}^{m}\left[\sum_{j=1}^{n_i} D_{ij}\ln(\widetilde{T}_{ij}^{k-1}) + \ln\left((-1)^{D_{i\bullet}} \cdot (-1)^{D_{i\bullet}} \frac{\lambda \cdot D_{i\bullet}!}{(\lambda + \frac{1}{k}\sum_{j=1}^{n_i} \widetilde{T}_{ij}^k)^{D_{i\bullet}+1}}\right)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n_i} D_{ij}(k-1)\ln(\widetilde{T}_{ij}) + \sum_{i=1}^{m}\left[\ln(\lambda) + \ln(D_{i\bullet})!) - (D_{i\bullet}+1)\ln\left(\lambda + \frac{1}{k}\sum_{j=1}^{n_i}\widetilde{T}_{ij}^k\right)\right] \\
&= (k-1)\sum_{i=1}^{m}\sum_{j=1}^{n_i} D_{ij}\ln(\widetilde{T}_{ij}) + m\ln(\lambda) + \sum_{i=1}^{m}\ln(D_{i\bullet}!) - \sum_{i=1}^{m}(D_{i\bullet}+1)\ln\left(\lambda + \frac{1}{k}\sum_{j=1}^{n_i}\widetilde{T}_{ij}^k\right).
\end{aligned}$$

To (try to) find expressions for the maximum likelihood estimators we need to find expressions for the partial derivatives of the log-likelihood function,

$$\begin{aligned}
\frac{\partial \ell}{\partial \lambda} &= \frac{m}{\lambda} - \sum_{i=1}^{m} \frac{D_{i\bullet}+1}{\lambda + \frac{1}{k}\sum_{j=1}^{n_i}\widetilde{T}_{ij}^k}, \\
\frac{\partial \ell}{\partial k} &= \sum_{i=1}^{m}\sum_{j=1}^{n_i} D_{ij}\ln(\widetilde{T}_{ij}) - \sum_{i=1}^{m}(D_{i\bullet}+1)\cdot \frac{-\frac{1}{k^2}\sum_{j=1}^{n_i}\widetilde{T}_{ij}^k + \frac{1}{k}\sum_{j=1}^{n_i}\widetilde{T}_{ij}^k\ln(\widetilde{T}_{ij})}{\lambda + \frac{1}{k}\sum_{j=1}^{n_i}\widetilde{T}_{ij}^k}.
\end{aligned}$$

We see that in both partial derivatives both λ and k appears in the denominator of the fraction inside the sum over i, so when equating them to zero we are not able to solve any of the two resulting equations with respect to any of the two parameters. To find the maximum likelihood estimates we therefore need to resort to numerical optimisation, for example the Newton-Raphson algorithm.

Problem 6

Since $Z_i \geq 0$ for all i we must have that $X(t) - X(s) \geq 0$ for all $t \geq s$. Thereby, for $t > s$,

$$\mathrm{E}[X(t)|\mathcal{F}_s] = \mathrm{E}[(X(t) - X(s)) + X(s)|\mathcal{F}_s] = \mathrm{E}[X(t) - X(s)|\mathcal{F}_s] + X(s) \geq X(s).$$

In a counting process at most one event can happen at the same time, so $dN(t) \in \{0, 1\}$. If $dN(t) = 0$ we clearly have $dX(t) = 0$, whereas if $dN(t) = 1$ we have $dX(t) = Z_{N(t)}$. We can write this more compactly as

$$dX(t) = dN(t) \cdot Z_{N(t)}.$$

Since $Z_{N(t)} \in \{0, 1\}$ we have $dN(t) \cdot Z_{N(t)} \in \{0, 1\}$, which gives that

$$\begin{aligned}
dX^{\star}(t) &= \mathrm{E}[dX(t)|\mathscr{F}_{t-}] = \mathrm{E}[dN(t) \cdot Z_{N(t)}|\mathscr{F}_{t-}] \\
&= P(dN(t) \cdot Z_{N(t)} = 1|\mathscr{F}_{t-}) \\
&= P(dN(t) = 1, Z_{N(t)} = 1|\mathscr{F}_{t-}) \\
&= P(dN(t) = 1|\mathscr{F}_{t-}) \cdot P(Z_{N(t)} = 1|\mathscr{F}_{t-}, dN(t) = 1).
\end{aligned}$$

The definition of the intensity process $\lambda(t)$ gives that $P(dN(t) = 1|\mathscr{F}_{t-}) = \lambda(t)dt$. When $\mathscr{F}_{t-}$ is given and we know that $dN(t) = 1$, we know the value of $Z_{N(t-)}$ and that $N(t) = N(t-) + 1$. Since the counting process $N(t)$ is independent of the Z_i chain, the Markov structure of $\{Z_i\}_{i=1}^{\infty}$ thereby implies that

$$\begin{aligned}
P(Z_{N(t)} = 1|\mathscr{F}_{t-}, dN(t) = 1) &= P(Z_{N(t)} = 1|Z_{N(t-)}) \\
&= \alpha \mathbb{I}(Z_{N(t-)} = 0) + (1 - \beta)\mathbb{I}(Z_{N(t-)} = 1) \\
&= \alpha(1 - Z_{N(t-)}) + (1 - \beta)Z_{N(t-)}.
\end{aligned}$$

Thereby we have

$$dX^{\star}(t) = (\alpha(1 - Z_{N(t-)}) + (1 - \beta)Z_{N(t-)})\lambda(t)dt.$$

To get the compensator we integrate the incremental process $dX^{\star}(t)$ from zero to t,

$$X^{\star}(t) = \int_0^t dX^{\star}(u)du = \int_0^t (\alpha(1 - Z_{N(u-)}) + (1 - \beta)Z_{N(u-)})\lambda(u)du.$$

Using that $Z_{N(u-)}$ is constant between two subsequent event times T_{i-1} and T_i we get

$$\begin{aligned}
X^\star(t) &= \left[\sum_{i=1}^{N(t-)} \int_{T_{i-1}}^{T_i} (\alpha(1 - Z_{N(u-)}) + (1-\beta) Z_{N(u-)}) \lambda(u) du\right] \\
&\quad + \int_{T_{N(t-)}}^{t} (\alpha(1 - Z_{N(u-)}) + (1-\beta) Z_{N(u-)}) \lambda(u) du \\
&= \left[\sum_{i=1}^{N(t-)} \int_{T_{i-1}}^{T_i} (\alpha(1 - Z_{i-1}) + (1-\beta) Z_{i-1}) \lambda(u) du\right] \\
&\quad + \int_{T_{N(t-)}}^{t} (\alpha(1 - Z_{N(u-)}) + (1-\beta) Z_{N(u-)}) \lambda(u) du \\
&= \left[\sum_{i=1}^{N(t-)} (\alpha(1 - Z_{i-1}) + (1-\beta) Z_{i-1}) \int_{T_{i-1}}^{T_i} \lambda(u) du\right] \\
&\quad + (\alpha(1 - Z_{N(u-)}) + (1-\beta) Z_{N(u-)}) \int_{T_{N(t-)}}^{t} \lambda(u) du \\
&= \left[\sum_{i=1}^{N(t-)} (\alpha(1 - Z_{i-1}) + (1-\beta) Z_{i-1}) (\Lambda(T_i) - \Lambda(T_{i-1}))\right] \\
&\quad + (\alpha(1 - Z_{N(t-)}) + (1-\beta) Z_{N(t-)}) (\Lambda(t) - \Lambda(T_{N(t-)})),
\end{aligned}$$

where $\Lambda(t) = \int_0^t \lambda(u) du$ is the integrated intensity process of the counting process $N(t)$.