TMA4285 Time series models
Solution to exercise 1, autumn 2014

Problem 2.1 in Wei (2006). We have

$$Z_t = \begin{cases} Y_t & \text{when } t \text{ even,} \\ Y_{t-1} & \text{when } t \text{ odd,} \end{cases}$$

where $\ldots, Y_{-4}, Y_{-2}, Y_0, Y_2, Y_4, \ldots$ are independent and identically distributed with $\mathrm{P}(Y_t = -1) = \mathrm{P}(Y_t = 1) = 1/2$.

a) The marginal distribution for all Z_t is clearly

$$F_{Z_t}(z) = \mathrm{P}(Z_t \leq z) = \begin{cases} 0 & \text{for } z < -1, \\ \frac{1}{2} & \text{for } z \in [-1, 1), \\ 1 & \text{for } z \geq 1. \end{cases}$$

Thus, the process is first order stationary.

b) To see that the process is not second order stationary, we can for example observe that when t is even we have

$$\mathrm{E}[Z_t Z_{t+1}] = \mathrm{E}[Y_t \cdot Y_t] = 1,$$

whereas when t is odd we have

$$\mathrm{E}[Z_t Z_{t+1}] = \mathrm{E}[Y_{t-1} Y_{t+1}] = \mathrm{E}[Y_{t-1}] \cdot \mathrm{E}[Y_{t+1}] = 0.$$

Thereby $F_{Z_t, Z_{t+1}} \neq F_{Z_{t+1}, Z_{t+2}}$, and the process is not second order stationary.

Problem 2.2 in Wei (2006). Assuming Z_t to be defined for all real t, we get

$$\mu_t = \mathrm{E}[Z_t] = \mathrm{E}[U] \sin(2\pi t) + \mathrm{E}[V] \cos(2\pi t) = 0$$

and

$$\begin{aligned} \gamma_{t,t+k} = \mathrm{E}[Z_t Z_{t+k}] &= \mathrm{E}[U^2] \sin(2\pi t) \sin(2\pi(t+k)) + \mathrm{E}[U]\mathrm{E}[V] \sin(2\pi t) \cos(2\pi(t+k)) \\ &+ \mathrm{E}[U]\mathrm{E}[V] \cos(2\pi(t+k)) \sin(2\pi t) + \mathrm{E}[V^2] \cos(2\pi t) \cos(2\pi(t+k)) \\ &= \sin(2\pi t) \sin(2\pi t + 2\pi k) + \cos(2\pi t) \cos(2\pi t + 2\pi k) \\ &= \sin(2\pi t)(\sin(2\pi t) \cos(2\pi k) + \cos(2\pi t) \sin(2\pi k)) \\ &+ \cos(2\pi t)(\cos(2\pi t) \cos(2\pi k) - \sin(2\pi t) \sin(2\pi k)) \end{aligned}$$

$$= \cos(2\pi k)(\sin^2(2\pi t) + \cos^2(2\pi t)) = \cos(2\pi k).$$

Thereby we see that μ_t is constant and $\gamma_{t,t+k}$ is constant as a function of t, so the process is covariance stationary. Whether Z_t is strictly stationary depends on the distributions of U and V. To see that the process Z_t is not strictly stationary in general we can observe for example that

$$Z_0 = V \quad \text{and} \quad Z_{\frac{1}{4}} = U.$$

A necessary (but not sufficient) requirement for Z_t to be strictly stationary is thereby that the distributions of U and V are identical. If U and V are both normal, Z_t is strictly stationary (remember that strict stationarity and covariance stationarity is equivalent for normal processes), but in general Z_t is not strictly stationary.

Problem 2.3 in Wei (2006). No solution is given here.

Problem 2.4 in Wei (2006). To show that the given ρ_k is a valid autocorrelation function we need to show that it is positive semidefinite, i.e. that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \alpha_j \rho_{t_i - t_j} = \sum_{i=1}^{n} \alpha_i^2 + 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} \alpha_i \alpha_j \rho_{t_i - t_j} \geq 0$$

for all $n \in \mathbf{N}$, $\alpha_1, \ldots, \alpha_n \in \mathbf{R}$ and all $t_1, \ldots, t_n \in \mathbf{Z}$.

The given function ρ_k is in fact not positive semi-definite. To show this it is sufficient to show that the above inequality is not fulfilled for one combination of values for n and $t_1, \ldots, t_n$. For some $n \in \mathbf{N}$ consider, $t_i = i, i = 1, \ldots, n$. The quantity of interest then becomes

$$v = \sum_{i=1}^{n} \alpha_i^2 + 2\phi \sum_{i=1}^{n-1} \alpha_i \alpha_{i+1}$$

$$= [\alpha_1, \alpha_2, \alpha_3, \alpha_4, \cdots, \alpha_{n-1}, \alpha_n] \begin{bmatrix} 1 & \phi & 0 & 0 & \cdots & 0 & 0 \\ \phi & 1 & \phi & 0 & \cdots & 0 & 0 \\ 0 & \phi & 1 & \phi & \cdots & 0 & 0 \\ 0 & 0 & \phi & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \phi \\ 0 & 0 & 0 & 0 & \cdots & \phi & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \vdots \\ \alpha_{n-1} \\ \alpha_n \end{bmatrix}.$$

The v is then non-negative for all α_i's if and only if the above $n \times n$ matrix is postive semi-definite. Moreover, the matrix is postive semi-definite if and only if all eigenvalues of the matrix is non-negative. The eigenvalues of the above uniform tri-diagonal matrix is explicitely known and given by

$$\lambda_k = 1 + 2\phi \cos\left(\frac{k\pi}{n+1}\right), k = 1, \ldots, n.$$

In the following we first focus on

$$\lambda_n = 1 + 2\phi \cos\left(\frac{n-1}{n} \cdot \pi\right),$$

and observe that by choosing n large enough we can clearly get λ_n arbitrarily close to

$$\lim_{n\to\infty} \lambda_n = 1 - 2\phi.$$

As $1 - 2\phi < 0$ when $\phi \in (1/2, 1)$, ρ_k is thereby not a legal autocorrelation function when $\phi \in (1/2, 1)$.

Next focusing on

$$\lambda_1 = 1 + 2\phi \cos\left(\frac{1}{n+1} \cdot \pi\right),$$

we correspondingly observe that by choosing n large enough we can clearly get λ_1 arbitrarily close to

$$\lim_{n\to\infty} \lambda_1 = 1 + 2\phi.$$

As $1 + 2\phi < 0$ when $\phi \in (-1, -1/2)$, ρ_k is neither a legal autocorrelation function when $\phi \in (-1, -1/2)$.

Note: The given formula for ρ_k is a legal autocorrelation function when $\phi \in [-1/2, 1/2]$. It is then in fact an autocorrelation function of an MA(1) process, which is defined and discussed in Section 3.2.1 in Wei (2006).