TMA4285 Time series models
Solution to exercise 2, autumn 2015

Problem 1. We have

$$Z_t = \sum_{j=1}^{t} a_j.$$

We get

$$\mu_t = \mathrm{E}[Z_t] = \sum_{j=1}^{t} \mathrm{E}[a_j] = 0$$

and

$$\sigma_t^2 = \mathrm{Var}[Z_t] = \sum_{j=1}^{t} \mathrm{Var}[a_j] = \sum_{j=1}^{t} \sigma^2 = t\sigma^2.$$

As σ_t^2 is not a constant, the process is not second order weakly stationary. As the process has zero mean, the autcovariance function becomes

$$\gamma(t,s) = \mathrm{Cov}[Z_t, Z_s] = \mathrm{E}[Z_t Z_s] = \mathrm{E}\left[\left(\sum_{j=1}^{t} a_j\right)\left(\sum_{i=1}^{s} a_i\right)\right] = \sum_{j=1}^{t}\sum_{i=1}^{s} \mathrm{E}[a_j a_i]$$

$$= \sum_{j=1}^{t}\sum_{i=1}^{s} \sigma^2 I(i=j) = \sum_{j=1}^{\min(t,s)} \sigma^2 = \min(t,s)\sigma^2.$$

Thus,

$$\rho(t,s) = \frac{\gamma(t,s)}{\sqrt{\gamma(t,t)\gamma(s,s)}} = \frac{\min(t,s)\sigma^2}{\sqrt{t\sigma^2 s\sigma^2}} = \frac{\min(t,s)}{\sqrt{ts}}.$$

Problem 2. We have

$$Z_t = -1.7 + a_t - 0.6a_{t-1} + 0.3a_{t-2}.$$

This gives

$$\mu_t = \mathrm{E}[Z_t] = -1.7$$

and

$$\gamma(t,t+k) = \mathrm{E}[(a_t - 0.6a_{t-1} + 0.3a_{t-2})(a_{t+k} - 0.6a_{t+k-1} + 0.3a_{t+k-2})]$$
$$= \begin{cases} (1+0.6^2+0.3^2)\sigma^2 = 1.45\sigma^2 & \text{for } k=0, \\ (-0.6-0.6\cdot 0.3)\sigma^2 = -0.78\sigma^2 & \text{for } k=1, \\ 0.3\sigma^2 & \text{for } k=2, \\ 0 & \text{for } k \geq 3. \end{cases}$$

Thus, the process is stationary with variance $\sigma^2 = \gamma(t,t) = 1.45\sigma^2$, and the autocorrelation function becomes

$$\rho_k = \begin{cases} 1 & \text{for } k = 0, \\ \frac{-0.78}{1.45} = -0.53 & \text{for } k = 1, \\ \frac{0.3}{1.45} = 0.21 & \text{for } k = 2, \\ 0 & \text{for } k \geq 3. \end{cases}$$

As we know $\rho_k = \rho_{-k}$ this defines the whole autocorrelation function.

Problem 3. To show that the specified ρ_k is not postive semi-definite it is sufficient to show that there exist real values $\alpha_1, \alpha_2, \alpha_3$ so that

$$[\alpha_1 \alpha_2 \alpha_3] \begin{bmatrix} \rho_0 & \rho_1 & \rho_2 \\ \rho_{-1} & \rho_0 & \rho_1 \\ \rho_{-2} & \rho_{-1} & \rho_0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} < 0$$

Inserting values for ρ_k in the above quadratic expression define

$$g(\alpha_1, \alpha_2, \alpha_3) = [\alpha_1 \alpha_2 \alpha_3] \begin{bmatrix} 1 & 0.8 & 0.1 \\ 0.8 & 1 & 0.8 \\ 0.1 & 0.8 & 0.1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

$$= \alpha_1^2 + \alpha_2^2 + \alpha_3^2 + 1.6\alpha_1\alpha_2 + 0.2\alpha_1\alpha_3 + 1.6\alpha_2\alpha_3.$$

We set $\alpha_3 = 1$ and thereafter, to minimise the resulting expression with respect to α_1, we set the corresponding derivative with respect to α_1 equal to zero, i.e.

$$\frac{\partial g(\alpha_1, \alpha_2, 1)}{\partial \alpha_1} = 2\alpha_1 + 1.6\alpha_2 + 0.2 = 0 \Leftrightarrow \alpha_1 = -(0.8\alpha_2 + 0.1).$$

Substituting this expression for α_1 into g above we get

$$\min_{\alpha_1} g(\alpha_1, \alpha_2, 1) = (0.8\alpha_2 + 0.1)^2 + \alpha_2^2 + 1 - 1.6(0.8\alpha_2 + 0.1)\alpha_2$$

$$-0.2(0.8\alpha_2 + 0.1) + 1.6\alpha_2$$
$$= 0.36\alpha_2^2 + 1.44\alpha_2 + 0.99.$$

By putting the derivative of this with respect to α_2 equal to zero we get,

$$\frac{\partial \min_{\alpha_1} g(\alpha_1, \alpha_2, 1}{\partial \alpha_2} = 2 \cdot 0.36\alpha_2 + 1.44 = 0 \Leftrightarrow \alpha_2 = -\frac{1.44}{2 \cdot 0.36} = -2.$$

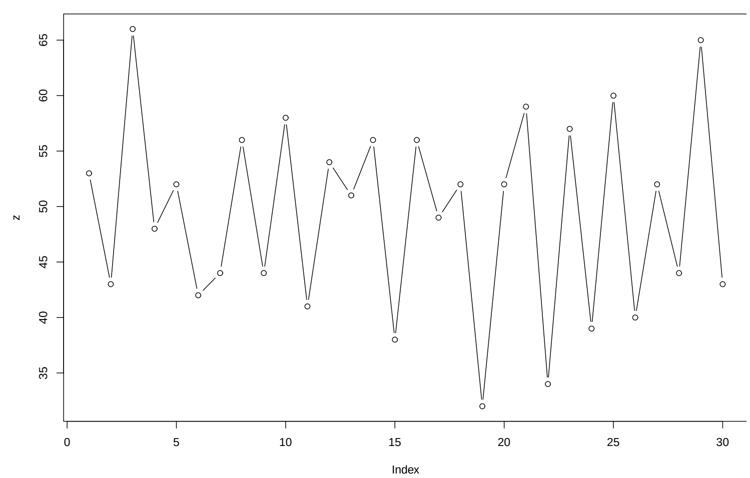
Insering this into the above expression for g we get

$$\min_{\alpha_1, \alpha_2} g(\alpha_1, \alpha_2, 1) = 0.36 \cdot (-2)^2 + 1.44 \cdot (-2) + 0.99 = -0.45.$$

Thus, we have shown that $g(1.5, -2, 1) < 0$, and thereby that ρ_k is not positive semi-definite.

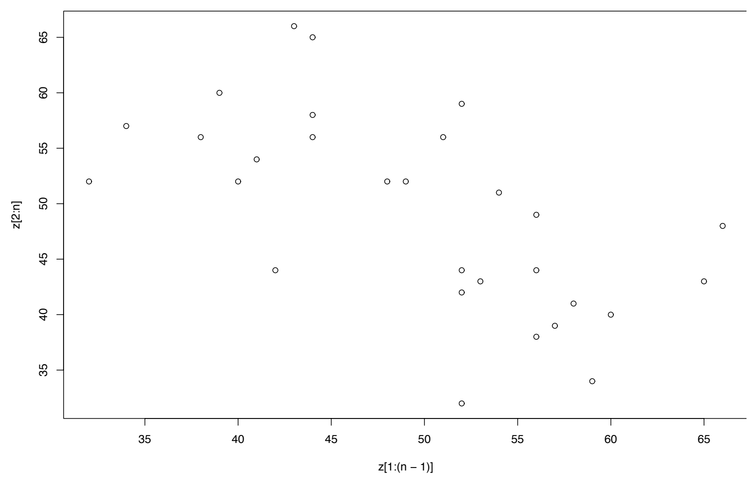
Problem 4.

a) Plot of the time series data:



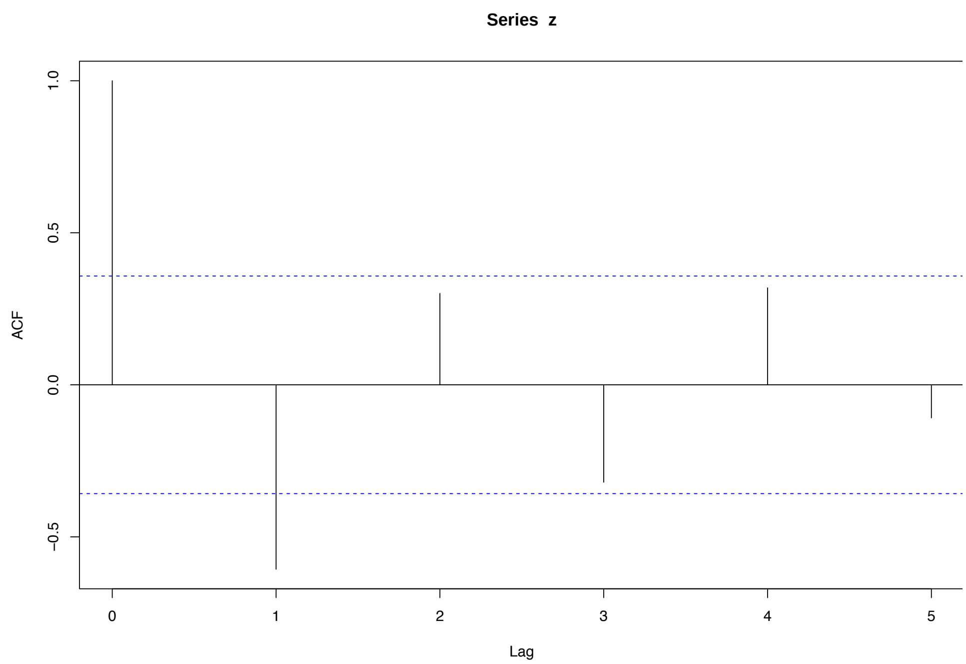
b) The first lag autocorrelation clearly seems to be negative, perhaps around -0.5.

c) Plot of z_t against z_{t+1}:



A value of approximately -0.5 for the first lag autocorrelation coefficient still seems reasonable.

d) Plot of estimated acf up to lag five:



e) Plot of estimated PACF up to lag five: