19.01.2016

Example

Normal distribution (Box-Muller)

Let $X_1 \sim U[0, 2\pi]$

$X_2 \sim Exp(\frac{1}{2})$ independently

$$\Rightarrow f_{\bar{X}}(x_1, x_2) = \frac{1}{2\pi} \cdot \frac{1}{2} e^{-\frac{1}{2}x_2} \qquad x_1 \in [0, 2\pi], \; x_2 \geq 0$$

Let $y_1 = \sqrt{x_2} \cdot \cos x_1$

$y_2 = \sqrt{x_2} \cdot \sin x_1$

This defines the function g where $y_1, y_2 \in \mathbb{R}$

Note $$y_1^2 + y_2^2 = x_2 \cdot \underbrace{(\cos^2 x_1 + \sin^2 x_1)}_{1} = x_2$$

$$\frac{y_2}{y_1} = \frac{\sin x_1}{\cos x_1} = \tan x_1 \Rightarrow x_1 = \tan^{-1}\left(\frac{y_2}{y_1}\right)$$

Thus

$$\left.\begin{aligned} y_1 &= \sqrt{x_2} \cos x_1 \\ y_2 &= \sqrt{x_2} \sin x_1 \end{aligned}\right\} \overset{\text{equivalent}}{\Longleftrightarrow} \left\{\begin{aligned} x_1 &= \tan^{-1}\left(\frac{y_2}{y_1}\right) \\ x_2 &= y_1^2 + y_2^2 \end{aligned}\right.$$

so the g-function is one-to-one

Then $$f_Y(y_1, y_2) = \frac{1}{2\pi} \cdot \frac{1}{2} \exp\left(-\frac{1}{2}(y_1^2 + y_2^2)\right) \cdot |J|$$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{1 + \left(\frac{y_2}{y_1}\right)^2} \cdot \left(-\frac{y_2}{y_1^2}\right) & 2y_1 \\ \frac{1}{1 + \left(\frac{y_2}{y_1}\right)^2} \cdot \frac{1}{y_1} & 2y_2 \end{vmatrix}$$

$$J = -\frac{y_2}{y_1^2+y_2^2}\cdot 2y_2 - \frac{y_1}{y_1^2+y_2^2}\cdot 2y_1$$

$$= -2\,\frac{y_1^2+y_2^2}{y_1^2+y_2^2} = -2$$

$$\Rightarrow \quad f_Y(y_1,y_2) = \frac{1}{2\pi}\cdot\frac{1}{2}e^{\left(-\frac{1}{2}(y_1^2+y_2^2)\right)}\cdot 2$$

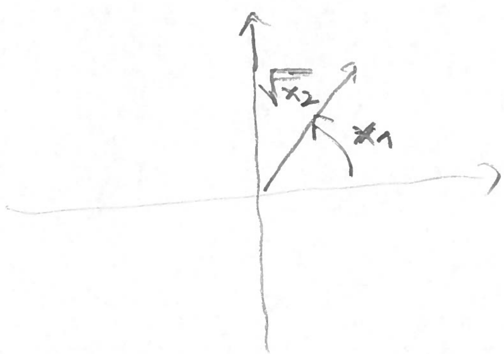
$$= \frac{1}{2\pi}e^{-\frac{1}{2}(y_1^2+y_2^2)}$$

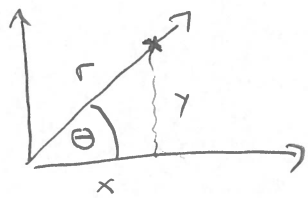
$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_1^2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y_2^2}$$

so that

$$y_1 \sim N(0,1) \qquad \text{independently}$$
$$y_2 \sim N(0,1)$$

<u>Graphical interpredation:</u>





$x = r\cdot\cos(\theta)$

$y = r\cdot\sin(\theta)$

$x_1 = \text{angle}$

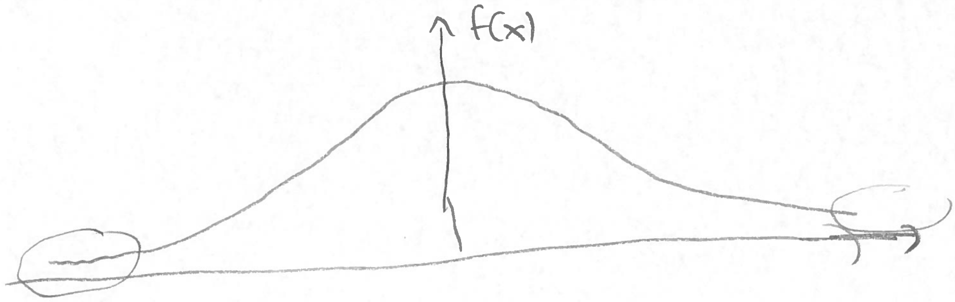
$\sqrt{x_2} = \text{radius}$

$\Rightarrow$ x_1, x_2 are polar coordinates while y_1 and y_2 are the Cartesian coordinates.

Example for ratio-of-uniforms method

A standard Cauchy distribution has density

$$f(x) = \frac{1}{\pi(1+x^2)} \; ; \qquad -\infty < x < \infty$$



Let $f^*(x) = \frac{1}{1+x^2}$ ($\frac{1}{\pi}$ would be the normalizing constant)

Then $\sqrt{f^*\left(\frac{x_2}{x_1}\right)} = \sqrt{\frac{1}{1+\left(\frac{x_2}{x_1}\right)^2}} = \sqrt{\frac{x_1^2}{x_1^2+x_2^2}} = x_1\sqrt{\frac{1}{x_1^2+x_2^2}}$

$$x_1 \leq \sqrt{f^*\left(\frac{x_2}{x_1}\right)}$$

$$\Updownarrow$$

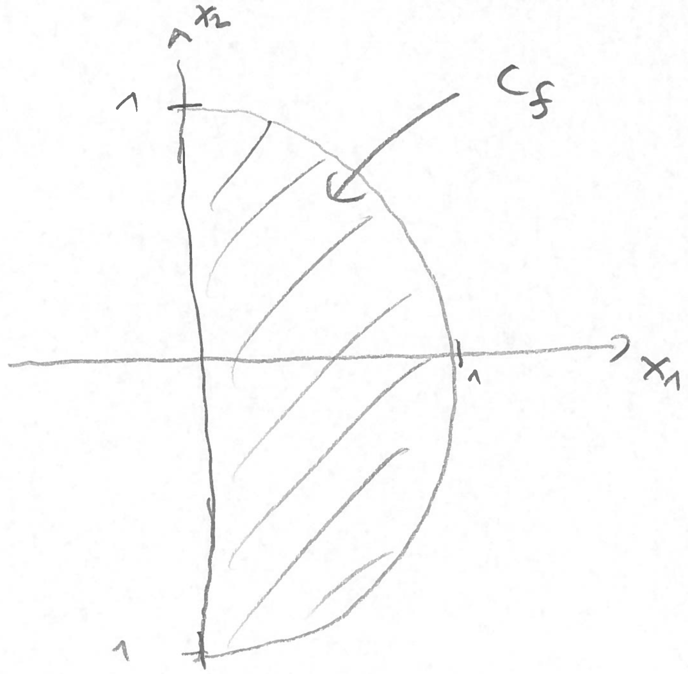
$$x_1 \leq x_1 \cdot \sqrt{\frac{1}{x_1^2+x_2^2}}$$

$$1 \leq \sqrt{\frac{1}{x_1^2+x_2^2}}$$

$$1 \leq \frac{1}{x_1^2+x_2^2}$$

$$x_1^2+x_2^2 \leq 1$$

$$C_f = \{(x_1,x_2) \mid x_1^2+x_2^2 \leq 1,\ x_1 \geq 0\}$$



$\hat{=}$ unit semi-circle

Proof of theorem:

a) C_f has finite area

$$\Delta_f = \iint_{C_f} 1 \, dx_1 dx_2 \underset{\uparrow}{=} \int_{-\infty}^{\infty} \int_0^{\sqrt{f^*(v)}} 1 \, \frac{\partial(x_1, x_2)}{\partial(u,v)} \cdot du\,dv$$

$$\left.\begin{aligned} u &= x_1 \\ v &= \frac{x_2}{x_1} \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} x_1 &= u \\ x_2 &= v \cdot u \end{aligned}\right.$$

$$\frac{\partial(x_1, x_2)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_2}{\partial u} \\ \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & v \\ 0 & u \end{vmatrix} = u \underset{0}{\overset{\wedge}{\geq}}$$

$$= \int_{-\infty}^{\infty} \int_0^{\sqrt{f^*(v)}} u \, du\,dv = \int_{-\infty}^{\infty} \frac{1}{2} u^2 \Big|_0^{\sqrt{f^*(v)}} dv$$

$$= \int_{-\infty}^{\infty} \frac{1}{2} f^*(v)\,dv = \frac{1}{2} \int_{-\infty}^{\infty} f^*(v)\,dv \underset{\text{by assumption}}{\leq} \infty$$

$\Rightarrow C_f$ has finite area

[Note using $f^*(v) = \frac{1}{1+v^2}$ we get $\Delta_f = \frac{1}{2} \cdot \pi$]

b) We know

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\underbrace{\frac{1}{2}\int_{-\infty}^{\infty} f^*(v)\,dv}_{\Delta_f}} \qquad (x_1,x_2) \in C_f$$

Consider the transformation

$$\left.\begin{aligned} u &= x_1 \\ v &= \frac{x_2}{x_1} \end{aligned}\right\} \iff \left\{\begin{aligned} x_1 &= u & 0 &\leq u \leq \sqrt{f^*(v)} \\ x_2 &= u \cdot v & -\infty &\leq v \leq \infty \end{aligned}\right.$$

$$f_{u,v}(u,v) = f_{x_1,x_2}(u, u\cdot v) \cdot |J|$$

$$= \frac{1}{\frac{1}{2}\int_{-\infty}^{\infty} f^*(v)\,dv} \cdot u$$

We are interested in the marginal density of $v = \frac{x_2}{x_1}$ i.e. we need to integrate out u.

$$\Rightarrow f_v(v) = \int_0^{\sqrt{f^*(v)}} f_{u,v}(u,v)\,du = \int_0^{\sqrt{f^*(v)}} \frac{u}{\frac{1}{2}\int_{-\infty}^{\infty} f^*(z)\,dz}\,du$$

$$= \frac{\frac{1}{2}u^2 \Big|_0^{\sqrt{f^*(v)}}}{\frac{1}{2}\int_{-\infty}^{\infty} f^*(z)\,dz} = \underbrace{\frac{\cancel{\frac{1}{2}} f^*(v)}{\cancel{\frac{1}{2}}\int_{-\infty}^{\infty} f^*(z)\,dz}}_{\text{normalized density}}$$

$$= f_V(v)$$