Lecture 3: inverse transform technique

Let F be a distribution, and let $U \sim \mathcal{U}[0,1]$.

a) Let F be the distribution function of a random variable taking non-negative integer values. The random variable X given by

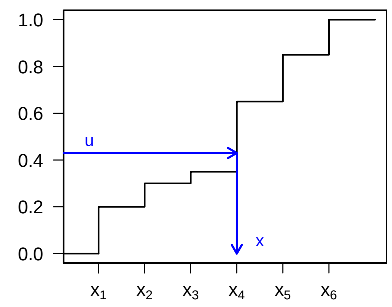
$$X = x_i \quad \text{if and only if} \quad F_{i-1} < u \leq F_i$$

has distribution function F.

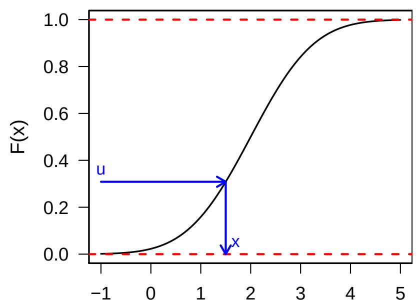
b) If F is a continuous function, the random variable $X = F^{-1}(u)$ has distribution function F.

Review: inverse transform technique (II)

a) Discrete case:



b) Continuous case:



The inverse transform technique is conceptually easy, but

- in the discrete case, a large number of comparisons may be necessary.
- in the continuous case, F^{-1} must be available.

Note

The inversion method cannot always be used! We must have a formula for $F(x)$ and be able to find $F^{-1}(u)$. This is for example not possible for the normal, χ^2, gamma and t-distributions.

Gamma distribution

Let $X \sim \text{Ga}(\text{shape}=\alpha, \text{rate}=\beta)$, i.e.

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta \cdot x}, x > 0.$$

From stochastic processes we know that if $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Exp}(\lambda)$, then $X_1 + \ldots + X_n \sim \text{Ga}(n, \lambda)$.

This gives how to simulate when α is an integer.

Gamma distribution

Further remember: $\chi^2_\nu = \text{Ga}(\frac{\nu}{2}, \frac{1}{2})$,

$X_1, \dots, X_n \overset{\text{iid}}{\sim} \mathcal{N}(0,1) \Rightarrow \sum_{i=1}^n X_i^2 \sim \chi^2_n$.

Thus, we can simulate $X \sim \text{Ga}(\frac{n}{2}, \frac{1}{2})$ by

$x = 0$
for $i = 1, 2, \dots, n$ **do**
  generate $y \sim \mathcal{N}(0,1)$ ▷ Still have to learn how
  $x \leftarrow x + y^2$
end for
return x

Gamma distribution (II)

β is a rate (inverse scale) parameter, i.e.

$$X \sim \text{Ga}(\alpha, 1) \quad \Leftrightarrow \quad X/\beta \sim \text{Ga}(\alpha, \beta)$$

Thus, we can simulate $X \sim \text{Ga}(\frac{n}{2}, \beta)$ by the algorithm

$x = 0$
for $i = 1, 2, \dots, n$ **do**
  generate $y \sim \mathcal{N}(0,1)$ ▷ Still have to learn how
  $x \leftarrow x + y^2$
end for
$x \leftarrow \frac{1}{2}x$ ▷ $\text{Ga}(\frac{n}{2}, 1)$
$x \leftarrow \frac{1}{\beta}x$ ▷ $\text{Ga}(\frac{n}{2}, \beta)$
return x

Thus, we know how to simulate from a $\text{Ga}(\alpha, \beta)$ whenever $\alpha = \frac{n}{2}$ where n is an integer.

Linear transformations

Many distributions have scale parameters, for example

$$X \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad \sigma X \sim \mathcal{N}(0, \sigma^2)$$
$$X \sim \text{Exp}(1) \quad \Leftrightarrow \quad \frac{1}{\lambda} X \sim \text{Exp}(\lambda)$$
$$X \sim \mathcal{U}[0,1] \quad \Leftrightarrow \quad \beta X \sim \mathcal{U}[0, \beta]$$

Adding a constant can also helping us in some situations

$$X \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad X + \mu \sim \mathcal{N}(\mu, 1)$$

and thereby

$$X \sim \mathcal{N}(0,1) \quad \Leftrightarrow \quad \sigma X + \mu \sim \mathcal{N}(\mu, \sigma^2)$$

Review scaling: Change of variables

$X \sim \text{Exp}(1)$. We are interested in $Y = \frac{1}{\lambda} X$, i.e. $y = g(x) = \frac{1}{\lambda} x$.

$$g^{-1}(y) = \lambda y \qquad \frac{dg^{-1}(y)}{dy} = \lambda$$

Application of the change of variables formula leads to:

$$f_Y(y) = \exp(-\lambda y)\lambda$$

It follows: $Y \sim \text{Exp}(\lambda)$.

Exercise: Check other transformations, we mentioned.

Bivariate techniques

Remember: If $(x_1, x_2) \sim f_X(x_1, x_2)$

$$\text{and } (y_1, y_2) = g(x_1, x_2)$$

$$\Updownarrow$$

$$(x_1, x_2) = g^{-1}(y_1, y_2)$$

where g is a one-to-one differentiable transformation. Then

$$f_Y(y_1, y_2) = f_X(g^{-1}(y_1, y_2))|\mathbf{J}|$$

with the determinant of the Jacobian matrix $\mathbf{J}$

$$|\mathbf{J}| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$\Rightarrow$ Multivariate version of the change-of-variables transformation

Bivariate techniques (II)

If we know how to simulate from $f_X(x_1, x_2)$ we can also simulate from $f_Y(y_1, y_2)$ by

$$(x_1, x_2) \sim f_X(x_1, x_2)$$

$$(y_1, y_2) = g(x_1, x_2)$$

Return (y_1, y_2).

Example: Normal distribution (Box-Muller)

see blackboard

Ratio-of-uniforms method

General method for arbitrary densities f known up to a proportionality constant.

Theorem

Let $f^\star(x)$ be a non-negative function with $\int_{-\infty}^{\infty} f^\star(x)dx < \infty$. Let $C_f = \{(x_1, x_2) \,|\, 0 \le x_1 \le \sqrt{f^\star\left(\frac{x_2}{x_1}\right)}\}$.

a) *Then C_f has a finite area*

Let (x_1, x_2) be uniformly distributed on C_f.

b) *Then $y = \frac{x_2}{x_1}$ has a distribution with density*

$$f(y) = \frac{f^\star(y)}{\int_{-\infty}^{\infty} f^\star(u)du}$$

Example: Standard Cauchy distribution

see blackboard

Algorithm to sample form a standard Cauchy

Generate (x_1, x_2) from $\mathcal{U}(C_f)$ ($\leftarrow$ How can we do this?)

$y = \frac{x_2}{x_1}$

return y.

Proof of theorem

see blackboard

How to sample from C_f?

We have $C_f = \{(x_1, x_2) \mid 0 \leq x_1 \leq \sqrt{f^\star\left(\frac{x_2}{x_1}\right)}\}$. If $f^\star(x)$ and $x^2 f^\star(x)$ are bounded we have

$$C_f \subset [0, a] \times [b_-, b_+], \qquad \text{with}$$

- $a = \sqrt{\sup_x f^\star(x)} > 0$
- $b_+ = \sqrt{\sup_{x \geq 0}(x^2 f^\star(x))}$
- $b_- = -\sqrt{\sup_{x \leq 0}(x^2 f^\star(x))}$

Proof: see blackboard

Use Rejection sampling to sample from C_f.