Simulation from discrete distributions

- Let x be a stochastic variable so that $x \in \{x_1, \dots, x_k\}$, and

$$P(x = x_i) = p_i \quad \text{where} \quad \sum_{i=1}^{k} p_i = 1$$

- Define $F_i = \sum_{j=1}^{i} p_j$ for $i = 0, 1, \dots, k$
- General simulation algorithm:

  $u \sim U[0,1]$
  for $i = 1, 2, \dots, k$ **do**
      if $u \in (F_{i-1}, F_i]$ **then**
          $x = x_i$
      end if
  end for

- Note:
  - can be used for any discrete distribution, but can be inefficient
  - more efficent search algorithm can make the algorithm faster
  - specialised algorithm for specific distrbutions: binomial, negative binomial, poisson

Simulation from continuous distributions

- Probability integral transform (inversion method)
- Let x have density $f(x), x \in \mathbf{R}$, $F(x) = \int_{-\infty}^{x} f(z)\mathrm{d}z$
- General simulation algorithm:

  $u \sim U[0, 1]$
  $x = F^{-1}(u)$
  return x
- Note:
  - can only be used when we can find a formula for $F^{-1}(u)$
  - specialised algorithms for specific distributions: gamma
  - easy to handle scale and location parameters

Bivariate transformation formula

- Result: Assume

$$(x_1, x_2) \sim f_x(x_1, x_2) \quad \text{(density)}$$

and

$$\begin{aligned} (y_1, y_2) &= g(x_1, x_2) \\ &\Updownarrow \\ (x_1, x_2) &= g^{-1}(y_1, y_2). \end{aligned}$$

Then

$$f_y(y_1, y_2) = f_x(g^{-1}(y_1, y_2)) \cdot |J|,$$

where

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

- Thus, if we are interested in a density $f_y(y_1, y_2)$ we need to find a density $f_x(x_1, x_2)$ and a one-to-one transformation $(y_1, y_2) = g(x_1, x_2)$ so that the above result holds true

Example: Standard normal (Box-Muller, 1958)

- Assume x_1 and x_2 independent and

$$x_1 \sim U[0, 2\pi] \quad , \quad x_2 \sim \text{Exp}\left(\frac{1}{2}\right).$$

- Thus,

$$f_x(x_1, x_2) = \frac{1}{2\pi} \cdot \frac{1}{2} e^{-\frac{1}{2}x_2} \quad \text{for } x_1 \in [0, 2\pi], x_2 > 0.$$

- Define

$$y_1 = \sqrt{x_2} \cos x_1 \qquad \text{and} \qquad y_2 = \sqrt{x_2} \sin x_1$$

$$\Updownarrow$$

$$x_1 = \tan^{-1}\left(\frac{y_2}{y_1}\right) \qquad \text{and} \qquad x_2 = y_1^2 + y_2^2$$