Markov chain Monte Carlo idea

- Situation:
  - Given a target distribution $f(x)$
  - Want to generate samples from $f(x)$

- Idea:
  - construct a Markov chain $\{X_i\}_{i=1}^{\infty}$ so that

$$\lim_{i \to \infty} P(X_i = x) = f(x)$$

  - simulate the Markov chain for many iterations
  - for m large enough $x_m, x_{m+1}, \ldots$ are (essentially) from $f(x)$

How to construct the Markov chain

- ⋆ How to construct such a Markov chain? ($x \in \Omega$ discrete)
  - Markov chain transition probabilities:
    $P(y|x) = P(X_{i+1} = y|X_i = x)$
  - Need to have

$$f(y) = \sum_{x \in \Omega} f(x) P(y|x) \quad \text{for all } y \in \Omega$$

  - Sufficient condition: Detailed balance condition

$$f(x) P(y|x) = f(y) P(x|y) \quad \text{for all } x, y \in \Omega$$

Metropolis–Hastings

- ⋆ Detailed balance condition:

$$f(x)P(y|x) = f(y)P(x|y) \quad \text{for all } x, y \in \Omega$$

- ⋆ Metropolis–Hastings setup for $P(y|x)$:

$$P(y|x) = Q(y|x)\alpha(y|x) \ \ \text{when } y \neq x$$

$$P(x|x) = 1 - \sum_{y \neq x} Q(y|x)\alpha(y|x)$$

where

- $Q(y|x)$ is a transition matrix (proposal kernel)
- $\alpha(y|x) = \min\left\{1, \frac{f(y)}{f(x)} \cdot \frac{Q(x|y)}{Q(y|x)}\right\}$ is an acceptance probability

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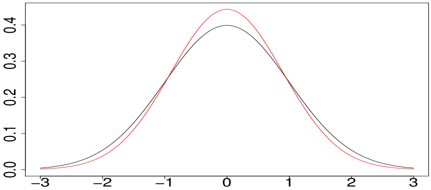
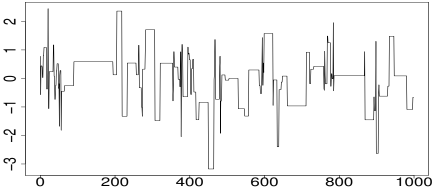
- ⋆ We have discussed two examples:
  - toy example: Poisson distribution
  - Ising model: x is a vector

Toy example: Independent proposals

- Target distribution: $x \sim \mathrm{N}_{250}(0, I)$
- Proposal distribution: $y|x \sim \mathrm{N}_{250}(0, 0.9^2 \cdot I)$

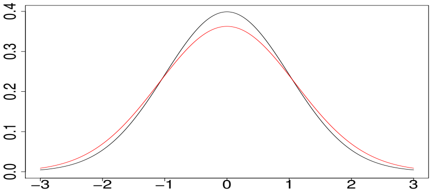


- Trace plot of x^1

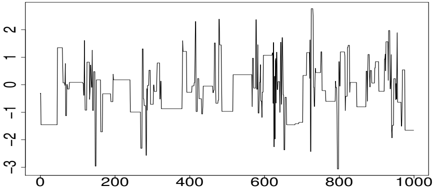


Toy example: Independent proposals

- ⋆ Target distribution: $x \sim \mathrm{N}_{250}(0, I)$
- ⋆ Proposal distribution: $y|x \sim \mathrm{N}_{250}(0, 1.1^2 \cdot I)$



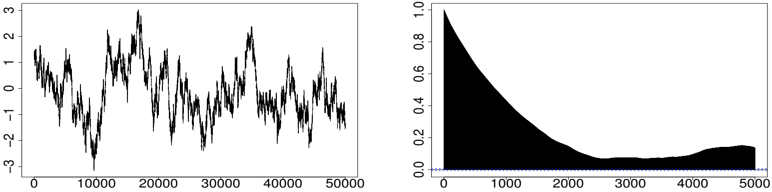
- ⋆ Trace plot of x^1



Toy example: Random walk proposals

- ⋆ Target distribution: $x \sim \mathrm{N}_{250}(0, I)$
- ⋆ Proposal distribution: $y|x \sim \mathrm{N}_{250}(x, \sigma^2 \cdot I)$
- ⋆ Trace plot and acf of x^1

$\sigma = 0.05$, acceptance rate $= 0.69$



Toy example: Random walk proposals