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TMA4305 PARTIAL DIFFERENTIAL EQUATIONS

Engelsk
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kl. 9–13

Hjelpemidler (kode C): Typegodkjent kalkulator med tomt minne (HP 30S), samt ett A4-ark stemplet av Institutt for matematiske fag, med valgfri påskrift av studenten.

Sensurdato: 21. juni 2007

All answers must be justified.

Problem 1 Determine the type (elliptic, parabolic or hyperbolic) of the equation

$$u_{xx} - 2u_{xy}\sin x - u_{yy}\cos^2 x - u_y \cos x = 0.$$

Find the characteristic curves (if there are any).

Problem 2 Solve by the method of characteristics:

$$x^2 u_x + y^2 u_y = u^2, \qquad u(x, 2x) = 1.$$

Problem 3 Let $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 < x < \pi,\ 0 < y < \pi\}$.

a) Suppose that $v \in C^2(\Omega)$ satisfies $v_{xx} + v_{yy} + xv_x + yv_y > 0$ in Ω. Prove that v has no local maximum in Ω.

b) Consider the problem

$$\begin{cases} u_{xx} + u_{yy} + xu_x + yu_y = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases}$$

where g is a given continuous function. Show that if $u \in C^2(\Omega) \cap C\big(\overline{\Omega}\big)$ is a solution, then the maximum of u is achieved on the boundary $\partial\Omega$.

Hint: Use the function $v_\varepsilon(x, y) = u(x, y) + \varepsilon x^2$.

Problem 4 An explosion generates a pressure wave satisfying the equation

$$P_{tt} - 16P_{xx} = 0 \qquad (x \in \mathbb{R},\ t > 0),$$

where $P(x,t)$ is the pressure at the point x and time t. The initial conditions at the explosion time $t = 0$ are

$$P(x,0) = \begin{cases} 10 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1, \end{cases}$$

$$P_t(x,0) = \begin{cases} 1 & \text{for } |x| \le 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

A building is located at the point $x_0 = 10$. The engineer who designed the building has determined that it will sustain a pressure up to $P = 6$. Find the time t_0 when the pressure on the building is maximal. Will the building collapse?

Problem 5 Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Assume that

$$u \in C^2\big(\Omega \times (0,\infty)\big) \cap C^1\big(\overline{\Omega} \times [0,\infty)\big)$$

is a solution of

$$\begin{cases} u_{tt} = c^2 \Delta u & \text{for } x \in \Omega,\ t > 0, \\ u(x,t) = 0 & \text{for } x \in \partial\Omega,\ t \ge 0, \end{cases}$$

where c is a constant. Show that the energy

$$\mathcal{E}(t) := \frac{1}{2} \int_\Omega \left(u_t^2 + c^2 |\nabla u|^2 \right) dx$$

is conserved (i.e., $\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \ge 0$).

Problem 6 Let $\Omega \subset \mathbb{R}^n$ (with $n \ge 2$) be a bounded domain with smooth boundary, and let $G(x,y)$ denote the Green's function for this domain. Use the maximum principle to show that

$$G(x,y) < 0 \qquad \text{for all } x, y \in \Omega \text{ with } x \ne y.$$

Problem 7 Let Ω be a bounded domain in $\mathbb{R}^n$. We assume that

$$x_1, \dots, x_n \geq 1 \qquad \text{for all } x = (x_1, \dots, x_n) \in \Omega.$$

As usual, we define $H_0^1(\Omega)$ as the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,2} = \left(\int_\Omega \left(|\nabla u|^2 + u^2 \right) dx \right)^{1/2}.$$

Let $f \in L^2(\Omega)$ be given. We define a functional $F : H_0^1(\Omega) \to \mathbb{R}$ by

$$F(u) = \int_\Omega \left(\sum_{i=1}^n \frac{1}{2} x_i^2 u_{x_i}^2 + f u \right) dx = \sum_{i=1}^n \frac{1}{2} \langle x_i u_{x_i}, x_i u_{x_i} \rangle + \langle f, u \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $L^2(\Omega)$ (i.e., $\langle g, h \rangle = \int_\Omega gh\, dx$).

a) Calculate $F(u+v) - F(u)$ for arbitrary $u, v \in H_0^1(\Omega)$. Find the Euler-Lagrange equation (the critical point equation) for F.

b) Prove that F is bounded below, i.e., there exists $M \in (0, \infty)$ such that $F(u) \geq -M$ for all $u \in H_0^1(\Omega)$.

c) By part (b), we know that

$$I := \inf_{u \in H_0^1(\Omega)} F(u)$$

is a well-defined real number. It can be shown (you do *not* have to do this here) that the minimum is attained, i.e., there exists $u \in H_0^1(\Omega)$ such that $F(u) = I$. This minimizer u is a weak solution of a partial differential equation (PDE). Which PDE are we talking about? Which boundary condition does u satisfy?