Some results on degradation processes with repairs and catastrophes

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Abstract

A degradation model based on a diffusion process and characterized by the presence of catastrophes and of repairs occurring according to a renewal process is presented. Explicit results are obtained under the assumption that the diffusive component of the degradation is a Wiener process with linear drift, and both catastrophe times and inter-repair times are exponentially distributed.

1 Outline of the model

A classical assumption in the theory of degradation models is that the quality of a device is described by a non-negative diffusion process $\{X(t),\, t \geq 0\}$ starting at $X(0) = x_0 > 0$. Assuming that large values of $X(t)$ correspond to a better quality, the level $X(t) = 0$ stands for zero quality, i.e. for the lack of functionality of the device. Consequently, the first-passage time T of $\{X(t)\}$ through state zero gives the failure time of the device. We shall denote its probability density function (p.d.f.) by

$$g(0,t\,|\,x_0) = \frac{\partial}{\partial t}\,\mathrm{P}\{T \leq t\}, \qquad t > 0. \tag{1}$$

In this note we assume that the quality of the device is improved by means of repairs due to maintenance, occurring as events of a renewal process $\{N(t),\, t \geq 0\}$ defined as

$$N(t) = \sum_{k=1}^{\infty} \mathbf{1}_{\{Y_1+\cdots+Y_k \leq t\}}, \qquad t \geq 0,$$

with $Y_1, Y_2, \ldots$ absolutely continuous non-negative i.i.d. random variables characterized by survival function $\overline{F}_Y(t) = \mathrm{P}(Y > t)$ and p.d.f. $f_Y(t)$. The effect of each repair is to reset instantaneously the quality of the device to the starting level x_0. Moreover, we assume that the device is also subject to random catastrophes whose effect is to inactivate it, i.e. to set instantaneously its quality to the zero level. In the following we shall denote by C the absolutely continuous random variable that describes the first occurrence of a catastrophe and by $\overline{F}_C(t)$ and $f_C(t)$ its survival function and p.d.f., respectively. We finally assume that C, $\{X(t)\}$ and $\{N(t)\}$ are independent.

Let us now denote by $\{\widetilde{X}(t),\, t \geq 0\}$, with $\widetilde{X}(0) = x_0 > 0$, the jump-diffusion process that describes the quality of the device in the presence of repairs and catastrophes according to the above assumptions. In Section 2 we shall obtain various results concerning the device failure time for model $\{\widetilde{X}(t)\}$. In Sections 3 and 4 these results are specialized, respectively, to the cases when both catastrophe times and inter-repair times are exponentially distributed and when, in addition, $\{X(t)\}$ is a Wiener process with linear drift.

2 Failure times

Referring to the model introduced in Section 1 we aim to study the device failure time, which is described by the first-passage-time random variable

$$\widetilde{T} = \inf\{t \geq 0 : \widetilde{X}(t) = 0\}, \qquad \widetilde{X}(0) = x_0 > 0.$$

For all $t > 0$ let us now set

$$\widetilde{g}(0,t\,|\,x_0) = \frac{\partial}{\partial t}\,\mathrm{P}\{\widetilde{T} \le t\}$$

and

$$\widetilde{g}_n(0,t\,|\,x_0) = \frac{\partial}{\partial t}\,\mathrm{P}\{\widetilde{T} \le t,\ N(t) = n\}, \qquad n = 0,1,\ldots\,. \tag{2}$$

We thus have

$$\widetilde{g}(0,t\,|\,x_0) = \sum_{n=0}^{+\infty} \widetilde{g}_n(0,t\,|\,x_0), \qquad t > 0. \tag{3}$$

In the following theorem we shall obtain the formal expression of functions $\widetilde{g}_n(0,t\,|\,x_0)$.

Theorem 1 *For all* $t > 0$ *we have*

$$\widetilde{g}_0(0,t\,|\,x_0) = -\overline{F}_Y(t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left[1-\int_0^t g(0,u\,|\,x_0)\,\mathrm{d}u\right]\overline{F}_C(t)\right\}, \tag{4}$$

$$\widetilde{g}_n(0,t\,|\,x_0) = \int_0^t \overline{F}_C(\theta)\left[1-\int_0^\theta g(0,u\,|\,x_0)\mathrm{d}u\right]\widetilde{g}_{n-1}(0,t-\theta\,|\,x_0)\,f_Y(\theta)\,\mathrm{d}\theta, \qquad n = 1,2,\ldots\,. \tag{5}$$

Proof. Density (4) is immediately obtained from identity

$$\widetilde{g}_0(0,t\,|\,x_0) = \left\{g(0,t\,|\,x_0)\,\overline{F}_C(t) + f_C(t)\left[1-\int_0^t g(0,u\,|\,x_0)\,\mathrm{d}u\right]\right\}\overline{F}_Y(t).$$

Conditioning on the instant $\theta \in (0,t)$ of the first repair and recalling (2), densities (5) easily follow. ■

Let us now introduce the Laplace transform of the failure-time p.d.f.:

$$\mathcal{F}(s) = \int_0^{+\infty} \mathrm{e}^{-st}\,\widetilde{g}(0,t\,|\,x_0)\,\mathrm{d}t, \qquad s \ge 0. \tag{6}$$

Herafter we obtain the expression of transform (6).

Theorem 2 *For* $s \ge 0$ *we have*

$$\mathcal{F}(s) = \mathcal{F}_0(s)\cdot\left\{1-\int_0^{+\infty}\mathrm{e}^{-st}\,f_Y(t)\left[1-\int_0^t g(0,u\,|\,x_0)\,\mathrm{d}u\right]\overline{F}_C(t)\,\mathrm{d}t\right\}^{-1}, \tag{7}$$

where

$$\begin{aligned}\mathcal{F}_0(s) = 1 &- s\int_0^{+\infty}\mathrm{e}^{-st}\,\overline{F}_Y(t)\left[1-\int_0^t g(0,u\,|\,x_0)\,\mathrm{d}u\right]\overline{F}_C(t)\,\mathrm{d}t\\ &-\int_0^{+\infty}\mathrm{e}^{-st}\,f_Y(t)\left[1-\int_0^t g(0,u\,|\,x_0)\,\mathrm{d}u\right]\overline{F}_C(t)\,\mathrm{d}t.\end{aligned} \tag{8}$$

Proof. Let us introduce the Laplace transform of $\widetilde{g}_n(0,t\,|\,x_0)$:

$$\mathcal{F}_n(s) = \int_0^{+\infty}\mathrm{e}^{-st}\,\widetilde{g}_n(0,t\,|\,x_0)\,\mathrm{d}t, \qquad s \ge 0, \qquad n = 0,1,\ldots\,. \tag{9}$$

From Eq. (5) we have

$$\mathcal{F}_n(s) = \mathcal{F}_{n-1}(s)\cdot\int_0^{+\infty}\mathrm{e}^{-st}\,f_Y(t)\left[1-\int_0^t g(0,u\,|\,x_0)\mathrm{d}u\right]\overline{F}_C(t)\,\mathrm{d}t, \qquad n = 1,2,\ldots\,. \tag{10}$$

Making use of Eqs. (3) and (9) the following holds:

$$\mathcal{F}(s) = \sum_{n=0}^{+\infty}\mathcal{F}_n(s), \qquad s \ge 0. \tag{11}$$

Hence, substituting (10) in (11) we easily obtain Eq. (7), whereas Eq. (8) follows from (4). ■

3 Exponential times

In this section we consider the special case when the inter-repair times $Y_1, Y_2, \ldots$ and the catastrophe time are exponentially distributed. Precisely, for $\lambda > 0$ and $\nu > 0$ we assume that

$$\overline{F}_Y(t) = \mathrm{e}^{-\lambda t}, \qquad \overline{F}_C(t) = \mathrm{e}^{-\nu t}, \qquad t \geq 0. \tag{12}$$

Let us now obtain the expression of the Laplace transform (6).

Theorem 3 *Under assumptions (12) we have*

$$\mathcal{F}(s) = 1 - \frac{s\left[1 - \mathcal{G}(s+\lambda+\nu)\right]}{s+\nu+\lambda\,\mathcal{G}(\lambda+s+\nu)}, \qquad s \geq 0, \tag{13}$$

where $\mathcal{G}(s) = \int_0^{+\infty} \mathrm{e}^{-st}\, g(0,t\,|\,x_0)\,\mathrm{d}t$, $s \geq 0$, *is the Laplace transform of density (1).*

Proof. Due to (12), Eq. (8) reads

$$\mathcal{F}_0(s) = \frac{\nu}{\lambda+s+\nu} + \frac{(s+\lambda)\,\mathcal{G}(s+\lambda+\nu)}{s+\nu+\lambda}. \tag{14}$$

Eq. (13) then follows from (7) and (14). ■

By virtue of Eqs. (6) and (13) we have

$$\int_0^{+\infty} \widetilde{g}(0,t\,|\,x_0)\,\mathrm{d}t = \mathcal{F}(s)\Big|_{s=0} = 1,$$

so that the failure of the device occurs w.p. 1. Moreover, it is

$$E(\widetilde{T}) = -\frac{\mathrm{d}\mathcal{F}(s)}{\mathrm{d}s}\Big|_{s=0} = \frac{1-\mathcal{G}(\lambda+\nu)}{\nu+\lambda\,\mathcal{G}(\lambda+\nu)}, \tag{15}$$

$$E(\widetilde{T}^2) = \frac{\mathrm{d}^2\mathcal{F}(s)}{\mathrm{d}s^2}\Big|_{s=0} = \frac{2\left[1-\mathcal{G}(\lambda+\nu)\right] + 2(\lambda+\nu)\,\mathcal{G}'(\lambda+\nu)}{\left[\nu+\lambda\,\mathcal{G}(\lambda+\nu)\right]^2}, \tag{16}$$

where

$$\mathcal{G}'(s) = \frac{\mathrm{d}\mathcal{G}(s)}{\mathrm{d}s} = -\int_0^{+\infty} t\,\mathrm{e}^{-st}\, g(0,t\,|\,x_0)\,\mathrm{d}t.$$

From Eqs. (15) and (16) we finally obtain

$$\mathrm{Var}(\widetilde{T}) = \frac{1+2(\lambda+\nu)\mathcal{G}'(\lambda+\nu)-\mathcal{G}^2(\lambda+\nu)}{\left[\nu+\lambda\mathcal{G}(\lambda+\nu)\right]^2}. \tag{17}$$

4 Wiener process

According to a customary hypothesis (see Kahle and Lehmann (1998), Kahle and Wendt (2000) or Di Crescenzo and Ricciardi (1996), for instance) we conclude this note by assuming that $\{X(t),\, t \geq 0\}$ consists in a Wiener process characterized by infinitesimal moments $A_1 = -c$ and $A_2 = \sigma^2$ ($c \geq 0$, $\sigma > 0$). In other words, the following stochastic equation holds:

$$\mathrm{d}X(t) = -c\,\mathrm{d}t + \sigma\,\mathrm{d}W(t), \qquad t \geq 0, \qquad X(0) = x_0, \tag{18}$$

where $\{W(t)\}$ denotes the standard Wiener process. As well known (see Ricciardi *et al.* (1999), for instance), the first-passage-time p.d.f. is given by

$$g(0,t\,|\,x_0) = \frac{x_0}{\sqrt{2\pi\sigma^2 t^3}}\exp\left\{-\frac{(x_0-ct)^2}{2\sigma^2 t}\right\}, \qquad t > 0$$

and its Laplace transform is (see Ricciardi (1977), for instance):

$$\mathcal{G}(s) = \exp\left\{\frac{x_0\,[c-\sqrt{c^2+2s\sigma^2}]}{\sigma^2}\right\}, \qquad s \geq 0. \tag{19}$$

By Eq. (13) it immediately follows that the Laplace transform of the failure-time p.d.f. is given by

$$\mathcal{F}(s) = \frac{\nu + (\lambda+s)\,\exp\left\{x_0\left[c-\sqrt{c^2+2\sigma^2(\lambda+s+\nu)}\right]/\sigma^2\right\}}{s+\nu+\lambda\,\exp\left\{x_0\left[c-\sqrt{c^2+2\sigma^2(\lambda+s+\nu)}\right]/\sigma^2\right\}}, \qquad s \geq 0.$$

Making use of the above assumptions we are now able to obtain the expectation and the variance of the failure time $\widetilde{T}$.

Theorem 4 *Under the assumptions (12) and (18) we have*

$$\begin{aligned} E(\widetilde{T}) &= \frac{1-\mathcal{E}}{\nu+\lambda\,\mathcal{E}}, \\ \mathrm{Var}(\widetilde{T}) &= \frac{\mathcal{E}}{\left(\nu+\lambda\,\mathcal{E}\right)^2}\left[\frac{1}{\mathcal{E}} - \mathcal{E} - \frac{2x_0\,(\lambda+\nu)}{\sqrt{c^2+2\sigma^2(\lambda+\nu)}}\right], \end{aligned} \tag{20}$$

where

$$\mathcal{E} = \exp\left\{x_0\left[c-\sqrt{c^2+2\sigma^2(\lambda+\nu)}\right]/\sigma^2\right\}.$$

Proof. Noting that (19) gives

$$\mathcal{G}'(s) = -\frac{x_0}{\sqrt{c^2+2s\sigma^2}}\exp\left\{\frac{x_0\,[c-\sqrt{c^2+2s\sigma^2}]}{\sigma^2}\right\}, \qquad s \geq 0,$$

the proof follows from Eqs. (15) and (17). ■

Finally we stress that the mean failure time (20) is decreasing if c increases, if σ increases and if ν increases, whereas it is increasing if x_0 increases and if λ increases, for large values of λ.

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