Bayesian Reliability-Growth Modeling of Repairable Mechanical Systems

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Abstract

This paper deals with the reliability analysis of a repairable mechanical system undergoing a development program in which modifications are introduced into the system design stage by stage in order to improve reliability. The system reliability is measured through the number of failures that will occur in a given time interval in a fleet of new systems and the failure pattern in each program stage is modeled by a PLP with increasing failure intensity. The use of a Bayesian procedure allows the system reliability to be estimated by combining failure data of current stage with data of previous stages and prior belief on the effectiveness of design changes. A numerical application is provided to illustrate the proposed procedure.

1 Introduction

Market requirements demand testing programs to be performed during the development phase of a repairable system in order to improve the equipment reliability until a prefixed reliability target is achieved.

Development programs are generally carried out in several stages. During each stage, one or more copies of the system are put on test and are minimally repaired at failure. At the end of each stage, the system reliability is estimated on the basis of test results and, if the reliability target is not achieved, the system design is changed by implementing corrective actions in the attempt to remove the observed failure causes (delayed fixes). Then, a new stage of testing begins. If the corrective actions are effective, the reliability of the new configuration is higher than that of the previous one. The development program is stopped when the reliability target is reached or a prefixed development time is expired (Calabria *et al.* 1996 and Ebrahimi 1996).

This paper proposes a reliability-growth model which analyses failure data from repairable mechanical systems subject to deterioration phenomena. Two different test scenarios are analyzed: *a*) *new* copies of the system are tested in each stage (they are built before starting a new stage by modifying the previous design), and *b*) the initial copies are tested during the whole program, so that each stage, except the first one, is performed on *used* systems (design modifications are then implemented on the *used* copies). In both cases, the number of copies can change with stages; of course, for test scenario *b*), such a number is a non-increasing sequence of values (some copies can be removed from testing).

For both scenarios, the failure process of the system between two successive design modifications is modeled by the Power-Law Process with increasing failure intensity: $\mu(t) = \lambda\,\beta\,t^{\beta-1}$ ($\beta > 1$). It is assumed that design modifications do not alter the failure mechanism, so that the time-trend parameter β remains unchanged with stages. The improvement in the failure intensity produced by design modifications is then modeled through a decrement of the parameter λ.

The system reliability, measured through the number of failures that will occur in a given time interval in a fleet of new units, is estimated by a Bayesian procedure, thus allowing both failure data of previous stages and prior information on the effectiveness of corrective actions to be introduced into the inferential procedure. This is particularly useful when the number of copies put on test and/or the number of observed failures is small, as for expensive prototype and/or testing.

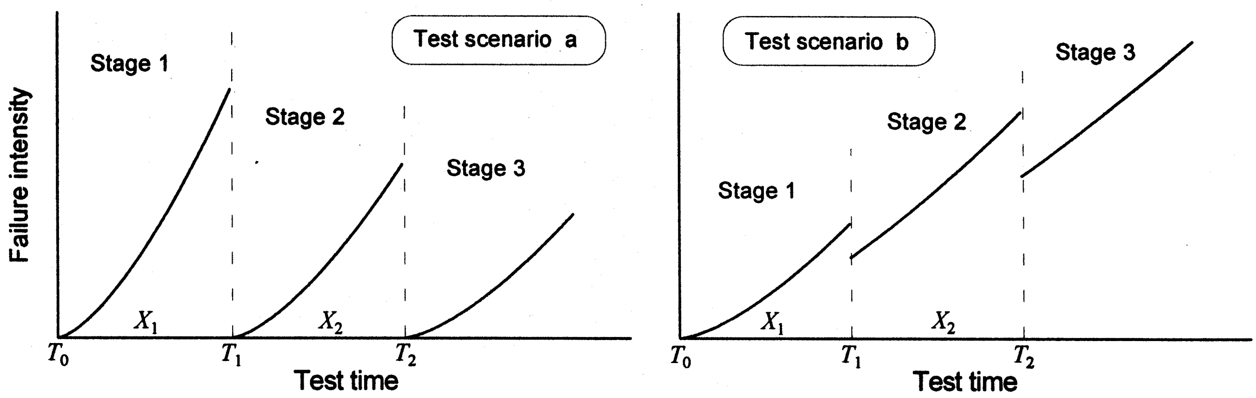

Figure 1: Test scenarios for development program

A vague prior density on model parameters is used at the first stage, whereas at each successive stage the prior density is obtained by combining the prior belief on the modification effectiveness and the posterior density at the previous stage. In this way, all failure data collected until the current stage are explicitly (through the likelihood function) or implicitly (through the prior density) used for making inference on the reliability of the current configuration.

2 Assumptions and model description

1. Testing procedure: k_j identical copies of the system are put on test at stage j ($j = 1, 2, \ldots$). Failures are observed until a prefixed test time X_j is accumulated. Two different test scenarios are analyzed (see Figure 1): *a*) *new* copies of the system are tested in each stage *b*) the initial copies are tested during the whole program, so that *used* copies are tested in each stage $j > 1$.

2. Failure process modeling: the failure process of each copy at stage j is modeled through a Power Law Process (PLP), whose increasing failure intensity is:

$$\mu_j(t) = \lambda_j \, \beta_j \, t^{\beta_j - 1} \quad \beta_j > 1, \; \lambda_j > 0 \tag{1}$$

The corrective actions do not alter the failure mechanism, so that $\beta_1 = \beta_2 = \ldots = \beta$. Thus, in order that $\mu_j(t) < \mu_{j-1}(t)$, we have that $\lambda_1 < \lambda_2 < \ldots$. Note that the operating time t is measured from the beginning of stage j, say T_{j-1}, for test scenario *a*), and from the beginning of the whole development program, say $T_0 = 0$, for test scenario *b*).

3. Reliability measure and stopping criterion: the system reliability is measured through the number $\mathcal{N}_\tau$ of future failures which will occur during the time period $(0, \tau)$ in a fleet of m new units. The development program is ended and the current system configuration is accepted for mass production when the probability that $\mathcal{N}_\tau$ is less than or equal to a given target $\mathcal{N}^*$ is no less than a prefixed value γ: $\Pr\{\mathcal{N}_\tau \leq \mathcal{N}^*\} \geq \gamma$.

4. Prior information at stage $j = 1$: No prior information is available, except that the system is experiencing deterioration, so that a noninformative prior density on β and λ_1 is used:

$$g(\beta, \lambda_1) \propto 1/(\beta^2 \lambda_1) \quad \beta > 1, \; \lambda_1 > 0 \tag{2}$$

5. Prior information at stage $j > 1$: A 3-parameter Gamma density (with unit location parameter) is used for β:

$$g_j(\beta) = \frac{b_j^{a_j} \exp(b_j)}{\Gamma(a_j)} (\beta - 1)^{a_j - 1} \exp(-b_j \beta) \qquad \beta > 1 \tag{3}$$

whose mean and variance are equal to the posterior mean $\mathrm{E}_{j-1}\{\beta|\text{data}\}$ and the posterior variance $\mathrm{V}_{j-1}\{\beta|\text{data}\}$, respectively, at the previous stage $j - 1$, so that:

$$a_j = \left[\mathrm{E}_{j-1}\{\beta|\text{data}\} - 1\right]^2 / \mathrm{V}_{j-1}\{\beta|\text{data}\} \qquad b_j = \left[\mathrm{E}_{j-1}\{\beta|\text{data}\} - 1\right] / \mathrm{V}_{j-1}\{\beta|\text{data}\}$$

A 2-parameter Gamma density is used for λ_j:

$$g(\lambda_j) = \frac{d_j^{c_j}}{\Gamma(c_j)} \lambda_j^{c_j-1} \exp(-d_j \lambda_j) \qquad \lambda_j > 0 \tag{4}$$

whose parameters c_j and d_j are obtained by combining the posterior estimates of λ_{j-1} with the prior knowledge on the modification effectiveness, measured through an improvement factor $\delta_j = \lambda_j/\lambda_{j-1} < 1$. The prior belief on the improvement factor δ_j is formalized by a Uniform density: $g(\delta_j) = 1/(\delta_{j,U} - \delta_{j,L})$ $(\delta_{j,L} \le \delta_j \le \delta_{j,U})$ and, under independence hypothesis, the prior mean $\mathrm{E}\{\lambda_j\}$ and second moment $\mathrm{E}\{\lambda_j^2\}$ of λ_j are equal to:

$$\mathrm{E}\{\lambda_j\} = \mathrm{E}\{\lambda_{j-1}|\text{data}\} \cdot \mathrm{E}\{\delta_j\} \quad \text{and} \quad \mathrm{E}\{\lambda_j^2\} = \mathrm{E}\{\lambda_{j-1}^2|\text{data}\} \cdot \mathrm{E}\{\delta_j^2\}$$

Thus: $c_j = [\mathrm{E}\{\lambda_j\}]^2/\mathrm{V}\{\lambda_j\}$ and $d_j = \mathrm{E}\{\lambda_j\}/\mathrm{V}\{\lambda_j\}$, where $\mathrm{V}\{\lambda_j\} = \mathrm{E}\{\lambda_j^2\} - [\mathrm{E}\{\lambda_j\}]^2$. The above assumptions aim at assuring that λ_j is stochastically smaller than $(\lambda_{j-1}|\text{data})$.

3 Mathematical developments

Let $t_{i,j,l}$ be the failure time i at stage j of unit l and let $n_{j,l}$ denote the number of failures experienced by unit l at stage j. Then, the likelihood function for the test data at stage j is:

$$L(\text{data}|\beta, \lambda_j) = \lambda_j^{N_j} \beta^{N_j} U_j^{\beta-1} \exp\left[-\lambda_j Z_j^{(\beta)}\right] \tag{5}$$

where $N_j = \sum_{l=1}^{k_j} n_{j,l}$ is the total number of failures at stage j, $U_j = \prod_{l=1}^{k_j} \prod_{i=1}^{n_{j,l}} t_{i,j,l}$ and

$$Z_j^{(\beta)} = \begin{cases} k_j X_j^\beta & \text{for test scenario } a) \\ k_j(T_j^\beta - T_{j-1}^\beta) & \text{for test scenario } b) \end{cases} \qquad \text{where } T_0 = 0 \text{ and } T_j = \sum_{i=1}^{j} X_i$$

Combining the likelihood (5) with the prior density (2), the posterior mean and second moment of β and λ_1 after the first stage result in:

$$\mathrm{E}_1\{\beta^r|\text{data}\} = \frac{1}{D_1} \int_1^\infty \beta^{N_1+r-2} U_1^{\beta-1} \left[Z_1^{(\beta)}\right]^{-N_1} d\beta \tag{6}$$

$$\mathrm{E}\{\lambda_1^r|\text{data}\} = \frac{\Gamma(N_1+r)}{\Gamma(N_1) \cdot D_1} \int_1^\infty \beta^{N_1-2} U_1^{\beta-1} \left[Z_1^{(\beta)}\right]^{-N_1-r} d\beta \tag{7}$$

where $D_1 = \int_1^\infty \beta^{N_1-2} U_1^{\beta-1} [Z_1^{(\beta)}]^{-N_1} d\beta$. By using now the informative prior densities (3) and (4), the posterior mean and second moment of β and λ_j after stage $j > 1$ result in:

$$\mathrm{E}_j\{\beta^r|\text{data}\} = \frac{1}{D_j} \int_1^\infty \beta^{N_j+r} (\beta-1)^{a_j-1} U_j^{\beta-1} \exp(-b_j\beta) \left[Y_j^{(\beta)}\right]^{-N_1-c_j} d\beta \tag{8}$$

$$\mathrm{E}\{\lambda_j^r|\text{data}\} = \frac{\Gamma(N_j+c_j+r)}{\Gamma(N_j+c_j) \cdot D_j} \int_1^\infty \beta^{N_1} (\beta-1)^{a_j-1} U_j{}^{\beta-1} \exp(-b_j\beta) \left[Y_j^{(\beta)}\right]^{-N_j-c_j-r} d\beta \tag{9}$$

where $Y_j^{(\beta)} = Z_j^{(\beta)} + d_j$ and $D_j = \int_1^\infty \beta^{N_j} (\beta-1)^{a_j-1} U_j^{\beta-1} \exp(-b_j\beta) [Y_j^{(\beta)}]^{-N_j-c_j} d\beta$.

If, at the end of stage j, the current configuration is accepted for mass production, then the number $\mathcal{N}_\tau$ of failures which occur in the time interval $(0, \tau)$ in a fleet of m new units is a Poisson random variable with mean value $\mathrm{E}\{\mathcal{N}_\tau\} = m\lambda_j\tau^\beta$:

$$\Pr\{\mathcal{N}_\tau = s\} = \frac{[m\lambda_j\tau^\beta]^s}{s!} \exp\left(m\lambda_j\tau^\beta\right) \qquad s = 0, 1, \ldots \tag{10}$$

By combining (10) with the likelihood (5) and the prior densities of β and λ_j, the posterior probability that $\mathcal{N}_\tau$ is no greater than $\mathcal{N}^*$ is given by:

$$\Pr\{(\mathcal{N}_\tau|\text{data}) \le \mathcal{N}^*\} = \begin{cases} \displaystyle\sum_{s=0}^{\mathcal{N}^*} \frac{\mathrm{B}(N_1, s)\, m^s}{s \cdot D_1} \int_1^\infty \beta^{N_1-2} U_1^{\beta-1} \tau^{s\beta} [Z_1^{(\beta)} + m\tau^\beta]^{-N_1-s} d\beta & j=1 \\ \displaystyle\sum_{s=0}^{\mathcal{N}^*} \frac{\mathrm{B}(N_j+c_j, s)\, m^s}{s \cdot D_j} \int_1^\infty \frac{\beta^{N_j}(\beta-1)^{a_j-1} U_j^{\beta-1} \tau^{s\beta} \exp(-b_j\beta)}{[Y_j^{(\beta)} + m\tau^\beta]^{N_j+c_j+s}} d\beta & j>1 \end{cases}$$

Table 1: Failure data in numerical application.

Stage 1				Stage 2				Stage 3				Stage 4		
#1	#2	#3	#4	#1	#2	#3	#4	#1	#2	#3	#4	#1	#3	#4
0.11	1.94	1.39	1.80	3.35	3.20	2.81	2.60	6.40	4.69	4.98	5.41	7.10	6.83	8.25
1.24	2.07	1.93	1.90	3.36	3.91	3.01	2.64	6.67	4.99		5.68	8.17	7.76	8.31
1.39	2.10	2.41		3.62	4.19	3.06	3.22		5.84		5.97	9.05	7.87	8.67
1.70	2.29			3.67	4.32	4.44	3.92		5.97		6.03		8.81	8.90
1.78				4.14							6.26			9.06
											6.37			
											6.41			
											6.66			
											6.74			

Table 2: Prior information and posterior inference in numerical application.

	Prior of β		Posterior of β		Prior of δ_j		Prior of λ_j		Posterior of λ_j		Posterior
j	Mean	SD	Mean	SD	$\delta_{j,L}$	$\delta_{j,U}$	Mean	SD	Mean	SD	$\Pr\{\mathcal{N}_\tau \leq 6\}$
1	Noninformative		1.777	0.457			Noninformative		0.734	0.346	0.487
2	1.777	0.457	1.666	0.241	0.5	0.9	0.514	0.260	0.567	0.230	0.650
3	1.666	0.241	1.634	0.162	0.5	0.9	0.397	0.176	0.382	0.137	0.868
4	1.634	0.162	1.617	0.120	0.6	1.0	0.306	0.119	0.300	0.094	0.946

4 Numerical application

The proposed procedure has been applied to the failure data of Table 1 which refer to test scenario *b*) with T_1=2.52, T_2=4.55, T_3=6.80 and T_4=9.16 u.t. The development program is performed initially on 4 units; then, after stage 3, the unit #2 is removed from testing. The number $\mathcal{N}_\tau$ of future failures is evaluated for $\tau = 1$ u.t. in a fleet of $m = 10$ units. The development program will be stopped when $\Pr\{(\mathcal{N}_\tau|\text{data}) \leq 6\} \geq 0.9$.

Table 2 gives the prior and posterior mean and standard deviation of β and λ_j, as well as the posterior probability to achieve the prefixed target $\mathcal{N}^* = 6$ failures. The program is stopped at stage 4 since the reliability target is reached. Figure 2 compares the posterior densities of β and λ_j for $j = 1, \ldots, 4$; we note that the uncertainty on β and λ_j decreases stage by stage.

References

Calabria, R., Guida, M. and Pulcini, G. (1996). A reliability-growth model in a Bayes-decision framework. *IEEE Transactions on Reliability 45* (3), 505–510.

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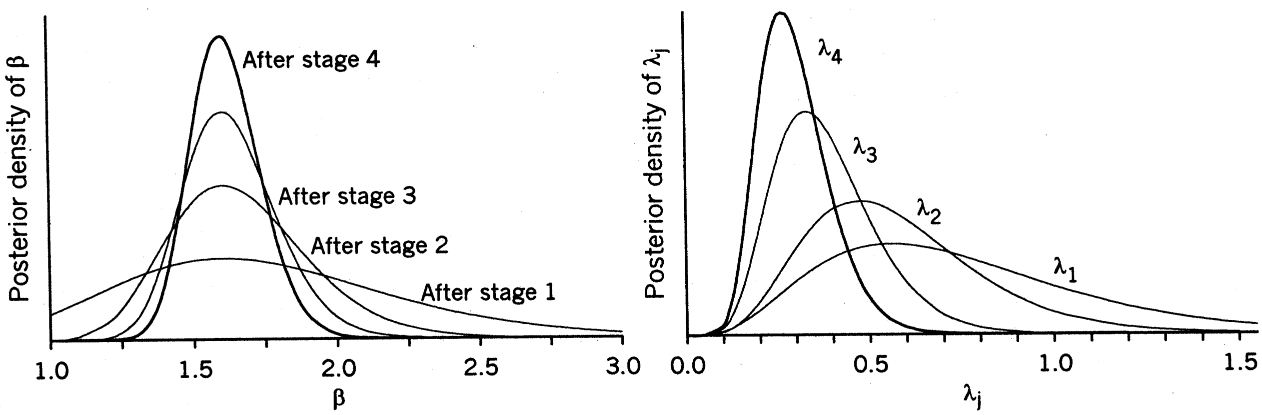

Figure 2: Posterior densities of β and λ_j after stages $j = 1, \ldots, 4$