Bayesian Prediction for the Shifted Exponential Distribution
Using Doubly Censored Data

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Abstract

On the basis of a doubly censored random sample of failure times drawn from a shifted exponential distribution, we address the problem of Bayesian prediction of the remaining observations of the sample (one-sample prediction), as well as the prediction of future observations from the same distribution (two-sample prediction). The Gibbs sampler is used to estimate the predictive distribution of the remaining testing time in the one-sample case and the total time on test up to a certain failure in a future sample. An example is used to illustrate the prediction procedure.

1 Introduction

Suppose that n components are put on test and that their lifetimes \(X_1, X_2, ..., X_n\) follow the shifted exponential distribution with location parameter \(\mu\) and failure rate \(\lambda\).

We consider the situation in which some observations are initially censored and where the life test is terminated before all items on test have failed. The resulting sample is said to be type II doubly censored. This type of censoring occurs in important practical situations (Nachlas and Kumar, 1993).

In most prediction problems related to the exponential distribution, the random variable of interest is an order statistic or a function of order statistics from the same or a future sample. Order statistics and their predictions do arise in the context of reliability and lifetesting where \(X_{(i)}\) represents the life length of an \((n-i+1)\)-out-of-\(n\) system made up of \(n\) identical components with independent life lengths.

In this paper, we present a Bayesian approach to predicting the behavior of further observations from the same distribution. Two prediction scenarios are considered; first, given the \((s - r + 1)\) observed lifetimes \(x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(s)}\) of a sample of \(n\) items, we predict the failure times of the remaining unfailed items \(x_{(s+1)}, x_{(s+2)}, ..., x_{(n)}\). This is referred to as one-sample prediction. The second scenario, known as the two-sample prediction, consists of predicting the first \(t\) failure times in a future sample of \(m\) components. Let us label the first case as Plan I and the second case as Plan II for future reference.

Several researchers have considered Bayesian prediction problems for the shifted exponential distribution. Dunsmore (1974) and Evans and Nigm (1980) discussed Bayesian prediction of future observations and uncensored sample observations based on a type II censored sample.

In Section 2, we use Gibbs sampling to provide sample-based estimates of the predictive density function of the total remaining testing time until all the items in the original sample have failed as well as that of the accumulated lifetimes of the \(m\) items in a future sample up to the time of the \(t^{th}\) failure. Section 3 includes an illustrative example using a real data set, and a discussion of the results.

2 Prediction of the Total Time on Test Using Gibbs Sampling

In the prior assessment, the exponential-gamma prior, used in Dunsmore (1974) and Madi & Leonard (1996), is used. Namely, we assume that given \(\lambda, \mu\) possesses the shifted positive exponential distribution with density

\[
\pi(\mu|\lambda) = c\lambda e^{-c\lambda(\eta - \mu)} f[\mu \leq \eta],
\]

where
where \( I[A] \) denotes the indicator function for the event \( A \) and where \( \eta \) is specified and \( \lambda \) has a gamma distribution with parameters \( \alpha \) and \( \beta \) with density function

\[
\pi(\lambda) \propto \lambda^{\alpha - 1} e^{-\beta \lambda}, \quad \lambda > 0.
\]

### 2.1 One-Sample Prediction Plan

Let us now consider the problem of predicting \( Y = (X_{(s+1)}, \ldots, X_{(n)}) \) \((r \leq s \leq n - 1)\) and then the remaining testing time, \( Z_1 = \sum_{i=s+1}^{n} X_{(i)} - (n - s) X_{(s)} \). The joint predictive density of \( Y \) does not allow us to obtain a closed expression for the predictive density of \( Z_1 \), therefore we opt for Gibbs sampling to estimate this latter density. Setting \( y_j = (x_{(s+1)}, \ldots, x_{(j-1)}, x_{(j+1)}, \ldots, x_{(n)}) \), the full conditional probability density function of \( X_{(j)} \) is found to be

\[
\pi(x_{(j)}|x, y_j, \mu, \lambda) = \begin{cases} 
\frac{\lambda e^{-\lambda x_{(j)}} I[x_{(j-1)} < x_{(j)} < x_{(j+1)}]}{e^{-\lambda x_{(j-1)}} - e^{-\lambda x_{(j+1)}}}, & j = s + 1, \ldots, n - 1 \\
\lambda e^{-\lambda x_{(n)}} I[x_{(n)} > x_{(n-1)}], & j = n
\end{cases}
\]

The full conditional distributions of \( \lambda \) and \( \mu \) are given by:

\[
\pi(\lambda|x, y, \mu) = \frac{\sum_{j=0}^{r-1} c_j \lambda^{n_r + \alpha} e^{-\lambda \Lambda_{1j}}}{\Gamma(n_r + \alpha + 1) \sum_{j=0}^{r-1} c_j \Lambda_{1j}^{-(n_r + \alpha + 1)}} \quad \text{and} \quad \pi(\mu|x, y, \lambda) = \frac{\sum_{j=0}^{r-1} c_j e^{-\lambda (x_{(i)} - \alpha_j \mu)} I[\mu < M_1]}{\sum_{j=0}^{r-1} c_j \lambda^{n_r} e^{-\lambda (x_{(i)} - \alpha_j \mu) M_1}}
\]

where \( c_j = (-1)^j (r-1) \), \( \alpha_j = n_r + c + j \), \( \kappa_1 \) is the arithmetic average of \( \alpha_j \)'s, \( \Lambda_{1j} = \sum_{i=0}^{n_r} x_{(i)} + \alpha_j (\kappa_1 - \mu) \) and for any 2 integers \( i \) and \( j \), \( j_i = j - i + 1 \). Given an arbitrary set of starting values of \( \mu, \lambda \), and \( y \), we generate values from (1) using the inverse cdf transformation method. The generation of values from (2) can be implemented based on the respective cdf's:

\[
F(\lambda) = \frac{\sum_{j=0}^{r-1} c_j \Lambda_{1j}^{-(n_r + \alpha + 1)} \gamma(n_r + \alpha + 1, \lambda \Lambda_{1j})}{\Gamma(n_r + \alpha + 1) \sum_{j=0}^{r-1} c_j \Lambda_{1j}^{-(n_r + \alpha + 1)}} \quad \text{and} \quad F(\mu) = \frac{\sum_{j=0}^{r-1} c_j e^{-\lambda (x_{(i)} - \alpha_j \mu)} M_1}{\sum_{j=0}^{r-1} c_j \alpha_j^{-(n_r + \alpha + 1)} e^{-\lambda (x_{(i)} - \alpha_j \mu) M_1}}
\]

where \( \gamma(a, x) = \int_0^x y^{a-1} e^{-y} \, dy \) is the incomplete gamma function.

### 2.2 Two-Sample Prediction Plan

In this plan, the problem of interest is that of predicting the total amount of testing time up to the \( T \) failure in a future sample of size \( m \) \((1 \leq t \leq m)\), \( Z_2 = \sum_{i=1}^{t} Y_{(i)} + (m - t) Y_{(t)} \). In the case where the failures are observed sequentially, we are interested in predicting the total time on test up to \( T \) failure, \( Z_3 = \sum_{i=1}^{t} Y_{(i)} \).

In case of simultaneous testing, Let \( Y_{(k)} = (y_{(1)}, \ldots, y_{(k-1)}, y_{(k+1)}, \ldots, y_{(t)}) \). It can be shown that the full conditional probability density function of \( y_{(k)} \) \((k = 1, \ldots, t)\) is

\[
\pi(y_{(j)}|x, y_{(j)}, \mu, \lambda) = \begin{cases} 
\frac{\lambda e^{-\lambda y_{(j)}} I[y_{(j-1)} < y_{(j)} < y_{(j+1)}]}{e^{-\lambda y_{(j-1)}} - e^{-\lambda y_{(j+1)}}}, & j = 1, \ldots, t - 1 \\
m_t e^{-m_t \lambda (y_{(t)} - y_{(t-1)})} I[y_{(t)} > y_{(t-1)}], & j = t
\end{cases}
\]

The full conditional distributions of \( \mu, \lambda \) are also obtained as

\[
\pi(\lambda|x, y, \mu) = \frac{\sum_{j=0}^{r-1} c_j \lambda^{n_r + \alpha} e^{-\lambda \Lambda_{2j}}}{\Gamma(s_r + \alpha + 1) \sum_{j=0}^{r-1} c_j \Lambda_{2j}^{-(s_r + \alpha + 1)}}
\]
and \( \pi(\mu | x, y, \lambda) = \frac{\sum_{j=0}^{r-1} c_j e^{-\lambda(j_{x(i)}-(\alpha_j+m))} I[\mu < M_2]}{\sum_{j=0}^{r-1} \lambda^{-1}(\alpha_j + m) - 1 e^{-\lambda(j_{x(i)}-(\alpha_j+m))} M_2} \),

where \( \Lambda_{2j} = \tau + z_2 + (\alpha_j + m)(\kappa_2 - \mu) \), \( \tau = \sum_{i=r} x(i) + (n-s)x(s) \) and \( \kappa_2 = \frac{\alpha_j c_j}{\alpha_j + m} \). In the case of sequential testing, the conditional probability density function \( \pi(y_k | x, y_k, \mu, \lambda) \), where \( y_k = (y_1, ..., y_{k-1}, y_{k+1}, ..., y_t) \), is shifted exponential distribution with parameters \( \mu \) and \( \lambda \). The full conditional distributions of \( \mu \), and \( \lambda \) are found to be

\[
\pi(\lambda | x, y, \mu) = \frac{\sum_{j=0}^{r-1} c_j \lambda^{s_r + t + \alpha} e^{-\lambda \Lambda_{3j}}}{\Gamma(s_r + t + \alpha + 1) \sum_{j=0}^{r-1} c_j \Lambda_{3j}^{-1}(s_r + t + \alpha + 1)},
\pi(\mu | x, y, \lambda) = \frac{\sum_{j=0}^{r-1} c_j e^{\lambda(\alpha_j + t)\mu} I[\mu < M_2]}{\sum_{j=0}^{r-1} \alpha_j \lambda^{-1}(\alpha_j + t) - 1 e^{\lambda(\alpha_j + t) M_2}}
\]

where \( \Lambda_{3j} = \tau + z_3 + (\alpha_j + t)(\kappa_3 - \mu) \).

3 Illustrative Example and Discussion

3.1 An Iterative Technique

The purpose of this section is to illustrate how the Gibbs sampler is used in predicting the future failure times (or missing observations) and make some prediction inferences about \( Z_1 \) or \( Z_2 \). After setting the initial values for \( \lambda, \mu \) and \( y \), say \( \lambda_0, \mu_0 \) and \( y_0 \), a Gibbs sampler single chain of 600 iterations is run and used as input in Raftery and Lewis Fortran program to determine the required number of iterations that should be run to attain convergence, (Raftery and Lewis 1992). Subsequent to convergence, 1000 draws of equally spaced variates were then collected for the 2 parameters \( \lambda \) and \( \mu \) as well as for the future failure times \( y \). The iterations for \( \lambda \) and \( \mu \) were drawn using their respective cdf’s using the inverse cdf technique, by means of the IMSL subroutines DRGNC3 and DRNGCT.

3.2 Illustrative Example

Consider the following Type II doubly censored data which represent the times to breakdown of an insulating fluid between electrodes recorded at the voltage 36 kV; Nelson (1982, pp.105) -\( r \), 0.96, 0.99, 1.69, 1.97, 2.07, 2.58, 2.71, 2.90, \(-r,-r,-r,-r\). Here, we have censored the 1st and 2nd observations as well as the last five failure times, \( n = 15, r = 3, s = 10 \). We wish to predict the remaining testing time, \( Z_1 \). The prior parameters were chosen to be \( \alpha = 2.0, \beta = 2.0, c = 1.0, \) and \( \eta = 2.0 \). Table 1 presents the values of mean, median, standard deviation (S.D.), and 2.5th and 97.5th percentiles of \( \mu, \lambda \) and \( Z_1 \). Figure 1 shows the estimate of the predictive density function of the remaining testing time estimated using the kernel approach, Silverman (1986). Bayesian inferences such as interval prediction can be performed. For example, the intervals \((-0.166, 0.813), (0.242, 0.713) \) and \((3.196, 27.559) \) provide 95% confidence and prediction intervals for \( \mu, \lambda \), and \( Z_1 \), respectively.

<table>
<thead>
<tr>
<th>Table 1: Summaries of the estimated distributions of ( \mu, \lambda ) and ( Z_1 )</th>
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<tbody>
<tr>
<td>Mean</td>
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<tr>
<td>( \mu )</td>
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<tr>
<td>( \lambda )</td>
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<tr>
<td>( Z_1 )</td>
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3.3 Discussion

We can get a feel on the performance of our predictors by comparing the predicted failure times to the actual observations. In our study, the Bayesian predictive estimates of the remaining failure times \( X(i) \), \( i = 11, ..., 15 \) are, respectively, 3.3472, 3.9284, 4.7129, 5.8128 and 8.097, compared with the true observations 3.67, 3.99, 5.35, 13.77 and 25.50. With the exception of the last 2 observations, which might be considered as extreme future failure times, the predictive estimates do fairly well.
Figure 1: Estimate of the predictive density function for $Z_1$ under Plan I

Experimentation with different starting values as well as with different values of the prior parameters led to the same results, which confirms convergence of the sampler and shows very good stability with respect to the prior settings.

Motivated by applications to reliability and life testing problems, many researchers have discussed various methods of prediction and used one- and two-parameter exponential parents as examples to illustrate these methods. Most prediction problems involve functions of order statistics from a current or a future sample. In this paper, we have tackled this prediction problem on the basis of a doubly censored random sample of failure times drawn from a shifted exponential parent. While the analytic approach to the development of the predictive distribution of $Z_1, Z_2$ and $Z_3$ is very complex, the Gibbs sampler proved to be an effective alternative iterative approach in simulating future observations which are used to estimate the predictive density of a function of these observations.

References