The reliability of quadratic dynamic systems

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Abstract

A key quantity for the assessment of the reliability of a dynamic system subjected to stochastic load processes is the mean level crossing rate of the response process. This paper discusses a new method for calculating this quantity for a stochastic response process represented as a second order stochastic Volterra series. The derivation of this procedure consists of three stages. First the expression for the mean crossing rate is rewritten in terms of a joint characteristic function. Secondly, it is shown that a closed form expression for this joint characteristic function can be derived. Thirdly, it is then demonstrated how the method of steepest descent can be applied to the numerical calculation of the mean crossing rate. It is shown by an example that the numerical accuracy of this method is apparently very high.

1 Introduction

From the point of view of practical assessment of the response statistics of engineering structures subjected to stochastic load processes, a quantity of particular importance is the average rate of upcrossings of high levels by the response. This is the key to e.g. estimation of extreme values.

In the present paper response processes will be considered that can be expressed as a second order stochastic Volterra series, that is, a stochastic Volterra series that has been truncated after the second order term. A substantial amount of work has been done to derive methods for efficient analysis of this model.

The type of stochastic Volterra series model that will be studied can be expressed as a sum of a linear and a nonlinear, quadratic transformation of a Gaussian process. Such a representation of the response process would apply to the standard model for expressing the total wave forces or horizontal excursion responses of e.g. a tension leg platform in a random sea way. Another example would be the response of a linear structure to a quadratic wind loading where the wind speed is modelled as a Gaussian process.

2 The Mean Crossing Rate

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $Z(t)$ be a real (strictly) stationary stochastic process with continuously differentiable sample paths (a.s.). It is assumed throughout that the distribution function of $Z(0)$, denoted by $F_Z(z) = P(Z(0) \leq z)$ is absolutely continuous. For every fixed level $\zeta \in R$, let $N_Z^+(\zeta)$ denote the rate of upcrossings of the level ζ by $Z(t)$, cf. Leadbetter, Lindgren, & Rootzen (1983), and let $\nu_Z^+(\zeta) = E[N_Z^+(\zeta)]$. Under the assumed conditions on $Z(t)$, it can be proved (Zähle 1984) that if $E[|Z(0)|] < \infty$, then

$$\nu_Z^+(\zeta) = E[\dot{Z}^+ | Z = \zeta] \, f_Z(\zeta) \tag{1}$$

where the equality holds a.s. (Lebesgue) with respect to ζ. $z^+ = \max(z, 0)$.

Assuming that the distribution of $(\dot{Z} | Z = \zeta)$ is absolutely continuous with a probability density function (PDF) $f_{\dot{Z}|Z}(s|\zeta)$, it follows that for a.e. ζ

$$\nu_Z^+(\zeta) = \int_0^\infty s \, f_{\dot{Z}|Z}(s|\zeta) \, ds \; f_Z(\zeta) = \int_0^\infty s \, f_{Z\dot{Z}}(\zeta, s) \, ds \tag{2}$$

where $f_{Z\dot{Z}}(\cdot,\cdot)$ denotes the joint PDF of $Z(0)$ and $\dot{Z}(0) = dZ(t)/dt|_{t=0}$. Equation (2) is often referred to as the Rice formula (Leadbetter et al. 1983). $\nu_Z^+(\zeta)$ is assumed throughout to be finite, and it is referred to as the mean upcrossing rate of the level ζ. With additional assumptions on the joint densities of $Z(0), \dot{Z}(0)$ and $Z(0), h^{-1}\big(Z(h)-Z(0)\big)$ for small values of h, it can be shown that equation (2) is valid for every ζ, cf. Marcus (1977) or Leadbetter, Lindgren, & Rootzen (1983). However, the required conditions to ensure equality in equation (2) for every value of ζ are not easy to verify, but most importantly, the stronger version is rarely needed. In reliability applications the critical levels for which the crossing rate is required can in general only be given with finite accuracy. This means that the crossing rate needs to be known for values of ζ belonging to small intervals who's length is determined by the level of the accuracy. Hence, the a.s. result is sufficient under such circumstances.

Denote the characteristic function of the joint variable $(Z, \dot{Z})$ by $M(\cdot,\cdot)$. Then $M(u, v) = E\,[\exp\big(iuZ + iv\dot{Z}\big)]$. Assuming that $M(\cdot,\cdot)$ is an integrable function, that is, $M(\cdot,\cdot) \in L^1(\mathbf{R}^2)$, it follows that (a.s.)

$$f_{Z\dot{Z}}(z, s) = \frac{1}{(2\,\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} M(u, v)\,\exp\big(-\,iuz - ivs\big)\,du\,dv \tag{3}$$

By substituting from equation (3) back into equation (2), the mean crossing rate is formally expressed in terms of the characteristic function, but this is not a very practical expression. It has been shown (Naess 2001; Naess 2002a) that a combination of equations (2) and (3) makes it possible to derive a more useful formula for the mean crossing rate expressed in terms of the characteristic function $M(u, v)$. Naess (2002a) shows that under suitable conditions on the characteristic function M, it is obtained that for a.e. ζ

$$\nu_Z^+(\zeta) = -\,\frac{1}{(2\,\pi)^2}\int_{-\infty}^{\infty}\left(⨍_{-\infty}^{\infty}\frac{1}{v}\,\frac{\partial M(u, v)}{\partial v}\,dv\right)e^{-\,iu\zeta}\,du \tag{4}$$

where the inner integral wrt v is interpreted as a principal value integral in the following sense: $⨍_{-\infty}^{\infty} = \lim_{\epsilon\to 0+}\big\{\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty}\big\}$.

3 The Response Process and its Joint Characteristic Function

It has been shown (Naess 1990; Naess 2002b) that for a wide class of second order stochastic Volterra series representing a system subjected to a stationary Gaussian process, the response $Z(t)$ can be represented as follows

$$Z(t) = \sum_{\alpha=1}^{n}\Big(\lambda_\alpha\,W_\alpha(t)^2 + c_\alpha W_\alpha(t)\Big) \tag{5}$$

where the λ_α and c_α are real constants, and the $W_\alpha(t)$ are continuously differentiable stationary (real) Gaussian processes. For each fixed t, the $W_\alpha(t)$ constitutes a set of independent $N(0, 1)$ random variables.

The derivation of an explicit expression for the characteristic function of the joint variable $(Z, \dot{Z})$ for $Z = Z(t)$ and $\dot{Z} = dZ(t)/dt$, will be based on the representation given by equation (5). For this purpose we introduce the Gaussian vector variables $W = (W_1, \ldots, W_n)'$, where $W_\alpha = W_\alpha(t)$, $\alpha = 1, \ldots, n$, and $\dot{W} = (\dot{W}_1, \ldots, \dot{W}_n)'$ (When X is a vector or a matrix, X' denotes the transpose of X). Then the Gaussian vector $(W', \dot{W}')'$ has covariance matrix

$$\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{pmatrix}, \tag{6}$$

where $\Sigma_{11} = \big(E[W_\alpha\,W_\beta]\big)$, $\Sigma_{12} = \big(r_{\alpha\beta}\big) = \big(E[W_\alpha\dot{W}_\beta]\big)$, $\Sigma_{21} = \big(E[\dot{W}_\alpha W_\beta]\big)$, and $\Sigma_{22} = \big(s_{\alpha\beta}\big) = \big(E[\dot{W}_\alpha\dot{W}_\beta]\big)$

In the present context, $\Sigma_{11} = I$, where I denotes the $n \times n$ identity matrix, and $\Sigma_{12}' = \Sigma_{21}$. It can also be shown that $r_{\alpha\beta} = E[W_\alpha\dot{W}_\beta] = -E[\dot{W}_\alpha W_\beta] = -\,r_{\beta\alpha}$, that is, $\Sigma_{12}' = -\,\Sigma_{12}$.

Naess (2001) has shown that the characteristic function of the joint variable $(Z, \dot{Z})$ is given by the expression

$$M_{Z\dot{Z}}(u,v) = \frac{1}{\sqrt{\det(A)}} \exp\Big(-\frac{1}{2}\, v^2\, c'\, V\, c + \frac{1}{2}\, t'\, A^{-1}\, t \Big) \tag{7}$$

where

$$t = (i\, u\, I + i\, v\, \Sigma_{12} - 2\, v^2\, D\, V)\, c \tag{8}$$

and

$$A = I - 2iuD \; - 2\, i\, v\, \big(D\, \Sigma_{21} + \Sigma_{12}\, D \big) + 4\, v^2 D\, V\, D \tag{9}$$

In the absence of the linear component, that is, when $Z(t) = \sum_{j=1}^{n} \lambda_j\, W_j(t)^2$, then

$$M_{Z\dot{Z}}(u,v) = \frac{1}{\sqrt{\det(A)}} \tag{10}$$

The marginal characteristic function $M_Z(u) = E[\, \exp\{i\, u\, Z\}\,]$ is now easily obtained by the relation $M_Z(u) = M_{Z\dot{Z}}(u,0)$. In particular, it is seen that $A = \mathrm{diag}(1 - 2iu\lambda_1, \ldots, 1 - 2iu\lambda_n)$. This gives $\det(A) = \prod_{j=1}^{n}(1 - 2\, i\, u\, \lambda_j)$ and $A^{-1} = \mathrm{diag}(d_1, \ldots, d_n)$, where $d_j = (1 - 2\, i\, u\, \lambda_j)^{-1}$. Hence, it is obtained that

$$M_Z(u) = \frac{1}{\sqrt{\prod_{j=1}^{n}(1 - 2\, i\, u\, \lambda_j)}} \exp\Big(-\frac{1}{2} \sum_{j=1}^{n} \frac{u^2\, c_j^2}{1 - 2\, i\, u\, \lambda_j} \Big) \tag{11}$$

which is in agreement with previously derived results, cf. e.g. Kac & Siegert (1947).

4 Numerical Calculation

The final step is the numerical calculation of the mean crossing rate by using equation (4) in combination with the expression for the joint characteristic function given by equation (7). However, a direct application of equation (4) for the numerical calculation turns out to pose severe problems due to the oscillatory term $e^{-iu\zeta}$. A way to alleviate this problem is to invoke the saddle point method, or the method of steepest descent. For this purpose equation (4) is rewritten as

$$\nu_Z^+(\zeta) = -\frac{1}{(2\,\pi)^2} \int_{-\infty}^{\infty} \frac{1}{v} \Big(\int_{-\infty}^{\infty} \exp\Big\{ -\; iu\zeta + \ln \frac{\partial M(u,v)}{\partial v} \Big\}\, du \Big)\, dv \tag{12}$$

which can be done for the case at hand. It is now observed that the function $g(\varpi) = -\, i\varpi\zeta + \ln \frac{\partial M(\varpi,v)}{\partial v}$ considered as a function of the complex variable ϖ is holomorphic in a region C, which includes the real line, of the complex plane $\mathbf{C}$. Assume that for each value of v and ζ, a saddle point $\varpi_s = \varpi_s(v,\, \zeta) = u_s - ib_s$ of the surface $\varpi \to |g(\varpi)|$ can be identified in C. If the path of steepest descent through the saddle point is approximately orthogonal to the imaginary axis, then, by shifting the path of integration from the real line to pass through the saddle point, a numerically stable and accurate estimate of the inner integral of equation (12) is usually obtained since along any path of steepest descent, the imaginary part of the function g remains constant,thus oscillations are avoided. If the path of steepest descent at the saddle point deviate from the direction orthogonal to the imaginary axis, then the integration line of the inner integral of equation (12) should be replaced by an integration path in C that follows the steepest descent path through the saddle point over a sufficient distance to achieve numerical accuracy, and which starts at $-\infty - i\, b_l$ and ends up at $\infty - i\, b_u$, where b_l and b_u are suitable constants. The direction of the steepest descent path from the saddle point is determined by the vector $-\,\overline{g'(\varpi)}$ for $\varpi \neq \varpi_s$ (Henrici 1977)

4.1 Example

To illustrate the accuracy of the numerical method, a simple model for the slow-drift response of a moored offshore structure is used. Specifically, the response process for this case may be written as ($\lambda = \lambda_1 = \lambda_2$) $Z_2(t) = \lambda\Big(W_1(t)^2 + W_2(t)^2\Big)$. For this process, the upcrossing rate $\nu^+_{Z_2}(\zeta)$ is given by the relation

$$\nu^+_{Z_2}(\zeta) = \frac{\hat{\sigma}_1}{\sqrt{2\pi}} \exp\Big(-\frac{\zeta}{2\lambda} + \frac{1}{2}\ln\big(\frac{\zeta}{\lambda}\big)\Big) \tag{13}$$

where $\hat{\sigma}_1 = \sqrt{s_{11} - (r_{12})^2}$. This special case provides a suitable test for the accuracy of the numerical method.

Let $\tilde{\nu}^+_{Z_2}(\zeta)$ denote the average upcrossing rate of $Z_2(t)$ calculated by the numerical method. Table 1 compares the analytical with the numerical upcrossing rate for different levels, and it is seen that the agreement is very good indeed.

Table 1: Comparison of analytical and numerical upcrossing rate.

z	$\nu^+_{Z_2}(\zeta)$	$\tilde{\nu}^+_{Z_2}(\zeta)$
0.5	$1.37 \cdot 10^{-2}$	$1.37 \cdot 10^{-2}$
1.0	$4.31 \cdot 10^{-3}$	$4.30 \cdot 10^{-3}$
2.0	$2.995 \cdot 10^{-4}$	$2.995 \cdot 10^{-4}$
3.0	$1.802 \cdot 10^{-5}$	$1.793 \cdot 10^{-5}$
5.0	$5.618 \cdot 10^{-8}$	$5.618 \cdot 10^{-8}$
7.0	$1.605 \cdot 10^{-10}$	$1.592 \cdot 10^{-10}$

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