Some results on residual entropy function

Jorge Navarro, Félix Belzunce, José M. Ruiz*
Dept Estadística e Investigación Operativa
Universidad de Murcia
30100 Murcia
Spain
jorgenav@um.es belzunce@um.es jmruizgo@um.es

Yolanda del Aguila
Dept. Estadística y Mat. Aplicada
Universidad de Almería
04120 Almería
Spain
yaguila@ual.es

Abstract

Ebrahimi and Pellerey (1995) and Ebrahimi (1996) proposed the Shannon residual entropy function as a useful dynamic measure of uncertainty. They studied the characterization problem from the residual entropy. They also used this function to define a stochastic order and two classes of distributions (DURL and IURL). In this paper, we obtain some new results on this function and we correct some mistakes in the preceding literature.

1 Introduction

The most used measures in Reliability associated to a positive random variable (r.v.) X with density f and reliability R, are the failure (hazard) rate $r(t) = f(t)/R(t)$ and the mean residual life $e(t) = E(X - t \mid X \geq t)$. It is well known that both r and e uniquely determine R (see Cox (1972) and Kotz and Shanbhag (1980)). Recently, Ebrahimi (1996) proposed another dynamic measured based on Shannon entropy. The Shannon entropy of X is defined by

$$H(X) = -E(\log f(X)) = -\int f(t) \log f(t) dt$$

It is a measure of the uncertainty and Ebrahimi considered the entropy of the residual variable, that is,

$$H(t) = H(X - t \mid X \geq t) = -\int_t^\infty \frac{f(x)}{R(t)} \log\left(\frac{f(x)}{R(t)}\right) dx,$$

as a dynamic measure of uncertainty. A similar definition can be given in the discrete case. Ebrahimi and Pellerey (1995) and Ebrahimi (1996) defined the uncertainty residual life order between two r.v. X and Y by

$$X \leq_{url} Y \Leftrightarrow H_X(t) \leq H_Y(t), \text{ for all } t \geq 0$$

They also defined the decreasing uncertainty residual life (increasing) class by

$$X \text{ is DURL (IURL) } \Leftrightarrow H_X(t) \text{ is decreasing (increasing)}$$

Ebrahimi (1996) showed that, in the absolutely continuous case, $H(t)$ uniquely determines $R(t)$. Rajesh and Nair (1998) gave an analogous result in the discrete case. Unfortunately, the proofs are not correct. In this paper, we show this mistake with some examples, giving conditions to obtain Ebrahimi's result. We also extend some recent characterization results given by Nair and Rajesh (1998) and Asadi and Ebrahimi (2000) based on relationships between H and e, and H and r, respectively. Moreover, we study ordering and classification properties based on residual entropy between a random variable X and the associated weighted r.v. Y with density

$$f_Y(t) = \frac{w(t)f(t)}{E(w(X))},$$

where w is a positive function such that $E(w(X)) < \infty$, extending the results given by Oluyede (1999) in the case $w(t) = t$.

*Supported by Ministerio de Ciencia y Tecnología under grant BFM2000-0362.

2 Direct characterization results

Theorem 2.1 If X is IURL, then $H(t)$ uniquely determines $R(t)$.

Proof. From the definition, we have

$$H'(t) = r(t)\,(H(t) - 1 + \log r(t)) \tag{1}$$

Hence, for a fixed $t > 0$ and $g(x) = x\,(H(t) - 1 + \log x) - H'(t)$, $r(t)$ is a positive solution of

$$g(x) = 0, \tag{2}$$

where $g(0) = -H'(t) \leq 0$, $g(+\infty) = +\infty$ and $g'(x) = H(t) + \log x$. Thus, g first decreases and then increases with a minimum at $x_0 = \exp(-H(t)) > 0$, which implies that (2) has a unique positive solution $r(t)$. So, $H(t)$ uniquely determines $r(t)$ and hence, $R(t)$.

Remark 2.1. Ebrahimi used the following reasoning. If r_1 and r_2 are two different failure rates with the same residual entropy H, and $r_1(t) \leq r_2(t)$ for a fixed t then, from (1),

$$H(t) - 1 + \log r_1(t) \geq H(t) - 1 + \log r_2(t) \tag{3}$$

holds, and hence $r_1(t) = r_2(t)$. However, $H'(t)$ can be negative and hence (3) does not hold.

Remark 2.2. Note that (2) has also a unique solution when $g(x_0) = 0$, which gives $H(t) = \log(b-t)$, i.e. the residual entropy for the Uniform distribution in $(0, b)$. Thus, the Uniform distribution can be characterized from a decreasing residual entropy $H(t) = \log(b - t)$.

Remark 2.3. If $H(t)$ is decreasing and $g(x_0) < 0$, then (2) has two positive solutions. However, one of the two solutions may be not a failure rate. Next, we see an example where both solutions are failure rates. If X has a Beta $\beta(c, 1)$ (Power) distribution $F(t) = t^c$ for $0 < t < 1$ and $c > 1$, then $r(t) = ct^{c-1}/(1 - t^c)$ and

$$H(t) = \frac{c-1}{c} + \log\left(\frac{1-t^c}{c}\right) + \frac{(c-1)t^c}{1-t^c}\log t$$

Thus, g has a minimum at

$$x_0 = \frac{c}{1-t^c} t^{-(c-1)t^c/(1-t^c)} \exp\{-\frac{c-1}{c}\},$$

which verifies

$$\frac{x_0}{r(t)} = t^{-(c-1)/(1-t^c)} \exp\{-\frac{c-1}{c}\} > 1$$

for $0 < t < 1$. So, (2) has two positive solutions $r_1 < x_0 < r_2$ and the Beta failure rate is $r(t) = r_1(t)$ for all t. Hence, the other solution $r_2(t)$, is also a failure rate function (i.e. $r_2 \geq 0$ and $\int_0^1 r_2 = \infty$).

Remark 2.4. In the discrete case, $H(t)$ we have an analogous result. In this case, we have a simple counterexample. If X is Bernoulli $B(p)$, then $H(0) = -p\log p - (1-p)\log(1-p)$ and $H(1) = 0$. Hence, $B(p)$ and $B(1-p)$ have the same residual entropy.

Remark 2.5. Analogous results can be obtained for the residual gamma entropy and/or the expected inactivity time entropy $H^*(t) = H(t - X \mid X \leq t)$.

3 Characterizations from relationships

Asadi and Ebrahimi (2000) characterized the generalized Pareto distribution (GPD) defined by $R(t) = (1+bt/a)^{-1/b}$, from the equality $H(t) = c - \log r(t)$, where c is a real-valued constant. This characterization includes the Exponential distribution ($c = 1$), the Pareto ($c > 1$) and the Finite Range ($c < 1$). In this section, we extend this result to the more general case where c is a function of t.

Theorem 3.1. If X has an absolutely continuous distribution $F(t)$ and $H(t) = c(t) - \log r(t)$, then

$$r(t) = \frac{1}{K - \int e^{c(t)}(1-c(t))dt} e^{c(t)}$$

Remark 3.1. In particular, if $c(t)$ is constant, then we obtain the characterization given by Asadi and Ebrahimi (2000) for the GPD which includes Exponential, Pareto and Finite Range distributions. Moreover, we can characterize new distribution models. For example, if $c(t) = at + b$ for $t > 0$ and $a > 0$, then we obtain the model with failure rate

$$r(t) = \frac{a}{Ke^{-at-b} + at + (b-2)}$$

We have to check if $r(t)$ is a failure rate (i.e. $r \geq 0$ and $\int_0^\infty r = \infty$) which in this case gives $K + (b-2)e^b \geq 0$.

In a similar way, Nair and Rajesh (1998) characterized the Exponential distribution from the equality $H(t) + e(t) = H(0) + e(0)$. They also characterized the type I Extreme Value distribution (defined by $F(t) = 1 - \exp(-\exp(t))$) from $H(t) + e(t) = 1 - t$. In the following theorem, we extend these results.

Theorem 3.2. If X has an absolutely continuous distribution $F(t)$, $H(t) + e(t) = c(t)$ and $c'(t) \geq -1$, then $c(t)$ uniquely determines $F(t)$.

Remark 3.2. In particular, we obtain the characterizations given by Nair and Rajesh (1998) for the Exponential and Extreme Value distributions when $c(t)$ is constant or $c(t) = 1 - t$.

4 Uncertainty in weighted distributions

Let Y be the weighted r.v. associated to X and a positive function $w(t)$. First, we study some necessary conditions to have the ordering $X \leq_{url} Y$ $(\geq)$.

Theorem 4.1. With the preceding notation,

i) If $E(w(X) \mid X \geq t)$ is increasing (decreasing), $E(w(X) \mid X \geq t)/w(t)$ is increasing (decreasing) and X is DFR (Y is DFR), then $X \leq_{url} Y$ $(\geq)$.

ii) If $w(t)$ is increasing (decreasing) and either X or Y is DFR, then $X \leq_{url} Y$ $(\geq)$.

The proof is based on Theorem 2.2 and 2.3 in Ebrahimi and Pellerey (1995) and the relationships between reliability measures of X and Y (see Jain et al. (1989)).

Remark 4.1. When Y is the length biased r.v. associated to X (i.e. $w(t) = t$), Oluyede (1999) proved that $X \leq_{url} Y$. Unfortunately, this result is not true (for example when X has a uniform distribution in $(0, 1)$). From preceding results, we have that a necessary condition to have Oluyede's result is that either X or Y is DFR.

Remark 4.2. When Y has the equilibrium distribution of a renewal process associated to X (i.e. $w(t) = 1/r(t)$), Ebrahimi and Pellerey (1995) gave a technical condition to have $X \leq_{url} Y$. We can extend this result from the preceding general results. Concretely, we have that the following conditions are equivalent

i) $X \leq_{url} Y$ $(\geq)$

ii) $\int_t^\infty f_X(x) \log r_X(x)dx \geq \frac{1}{e_X(t)} \int_t^\infty \overline{F}_X(x) \log(1/e_X(x))dx$ $(\leq)$.

iii) $\frac{1}{e_X(t)} \int_t^\infty \overline{F}_X(x) \log \overline{F}_X(x)dx - \int_t^\infty f_X(x) \log f_X(x)dx \leq \overline{F}_X(t) \log e_X(t)$ $(\geq)$.

In particular, we have that X is IMRL, implies $X \leq_{url} Y$.

Next, we study the necessary conditions to obtain the preservation of DURL and IURL classes.

Theorem 4.2. If $E(w(X) \mid X \geq t)$ is increasing (decreasing), $E(w(X) \mid X \geq t)/w(t)$ is decreasing (increasing) and $\lim_{t\to\infty} w(t) < \infty$ (> 0), then X is $DURL \Rightarrow Y$ is $DURL$ $(\Leftarrow)$

The proof is based on lemma 2.1 in Asadi and Ebrahimi (2000).

Theorem 4.3. If $E(w(X) \mid X \geq t)$ is increasing (decreasing), $E(w(X) \mid X \geq t)/w(t)$ is increasing (decreasing) and $\lim_{t\to\infty} w(t) < \infty$ (> 0), then X is $IURL \Rightarrow Y$ is $IURL$ $(\Leftarrow)$

Moreover, we have obtained another preservation result.

Theorem 4.4. If $X \leq_{fr} Y$ and $X \geq_{url} Y$, then

i) X is $DURL \Rightarrow Y$ is $DURL$

ii) Y is $IURL \Rightarrow X$ is $IURL$

The proof is based on equality (1).

Remark 4.3. When Y is the length biased r.v. associated to X (i.e. $w(t) = t$), from preceding results we have that if $X \geq_{url} Y$ holds, then (i) and (ii) in theorem 4.4 hold.

Remark 4.4. When Y has the equilibrium distribution of a renewal process associated to X (i.e. $w(t) = 1/r(t)$), the conditions obtained from the preceding results to obtain the preservation of DURL and IURL classes are obvious because $r_y(t) = 1/e_X(t)$ and $IFR(DFR) \subset DMRL(IMRL) \subset DURL(IURL)$.

5 References

Asadi, M. and Ebrahimi, N. (2000). Residual entropy and its characterizations in terms of hazard function and mean residual life function. *Statist. Probab. Lett. 29*(3), 241–288.

Cox, D. R. (1972). Regression models and life-tables. *J. Royal Statistical Soc. Ser B 34*, 187-202.

Ebrahimi, N. (1996). How to measure uncertainty about residual life time. *Sankhya Ser. A 58*, 48-57.

Ebrahimi, N. and Pellerey, F. (1995). New partial ordering of survival functions based on notion of uncertainty. *J. Appl. Probab. 32*, 202-211.

Jain, K., Singh, H. and Bagai, I. (1989). Relations for reliability measures of weighted distributions. *Comm. Statist. Theory Methods 18*(12), 4393-4412.

Kotz, S. and Shanbhag, D.N. (1980). Some new approaches to probability distributions. *Adv. Appl. Probab. 12*, 903-921.

Nair, K.R.M. and Rajesh, G. (1998). Characterization of probability distributions using the residual entropy function. *J. Indian Statist. Assoc. 36*, 157-166.

Oluyede, B.O. (1999). On inequalities and selection of experiments for length biased distributions. *Probab. Engineering Inform. Sci. 13*, 169-185.

Rajesh, G. and Nair, K.R.M. (1998). Residual entropy function in discrete time. *Far. East J. Theo. Statist. 2*(1), 43-57.