Asymptotic law of failure time for large systems with independent components

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Abstract

In this talk we present a Central Limit Theorem for the first failure time of generalized k-out-of-n system. Simulations for three different models are given to illustrate this result.

1 Introduction

We consider a system made of n i.i.d. Markovian components. Each component takes values into a finite state space E and starts from the same initial state e_0 (the perfect state). Let f be a degradation function defined on E with values in $\mathbb{R}$: for each e in E, $f(e)$ measures the level of degradation of a component in state e.

We assume that the system in state $\eta \in E^n$ is down if and only if the total degradation $\sum_{i=1}^n f(\eta_i)$ is larger than some threshold $k(n)$. Such systems can be viewed as a generalization of the so-called k-out-of-n systems.

Let $m(t)$ (respectively: $v(t)$) be the expectation (respectively: the variance) of the degradation at time t of one component starting from the perfect state at time 0. We assume that m is increasing on $[0;\tau[$ where $\tau \leq \infty$. Set $m_\star = m(0)$ and $m^\star = m(\tau)$. The following threshold is considered:

$$k(n) = n\alpha + o(\sqrt{n}) \quad \text{where} \quad \alpha \in]m_\star; m^\star[\ . \tag{1}$$

Denote by $X_1, \dots, X_n$ the states of components. The failure time of the system is the following:

$$T_n = \inf \left\{ t \geq 0 \ ; \ \sum_{i=1}^n f(X_i(t)) \geq k(n) \right\} . \tag{2}$$

In section 2 we present the Central Limit Theorem for T_n (theorem 1) and give a sketch some indications of its proof. The complete proof can be found in (Paroissin and Ycart 2001). Section 3 is devoted to simulations illustrating theorem 1: three different models is studied.

2 Main result

Since we have a Central Limit Theorem for finite-dimensional marginals of the sequence of i.i.d. processes $f(X_1), \dots, f(X_n)$, it is natural to hope the following Central Limit Theorem for the associated hitting time T_n:

Theorem 1 *Under the assumptions of section 1,*

$$\sqrt{n}(T_n - t_\alpha) \xrightarrow[n \longrightarrow +\infty]{\mathcal{L}} \mathcal{N}(0, \sigma^2) \ , \tag{3}$$

with:

$$t_\alpha = \inf\{t \ ; \ m(t) = \alpha\} \ . \tag{4}$$

and:

$$\sigma^2 = \frac{v(t_\alpha)}{(m'(t_\alpha))^2} \ . \tag{5}$$

Sketch of the proof First we prove that the sequence of processes $(Z_n(t))$ defined by:

$$Z_n(t) = \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{n} f(X_i(t)) - nm(t) \right) \ , \tag{6}$$

converges in distribution to a centered Gaussian process Z (for which the covariance function is explicitly known). In a second (and independent) part we prove an exponential inequality for the left tail of T_n. Then in a third part we consider the sequence of processes $(Z_n(t))$ renormalized in time (around the critical time t_α) and we prove that this sequence converges in distribution to a process with constant trajectories. This convergence allows us to conclude about the Central Limit Theorem for T_n.

3 Reliability models and simulations

In this section we present three reliability models. The first two deal with binary components while the third one is based on multi-state components. For each models we ran 100 trajectories of system made of 500 i.i.d. components. Thus we obtain samples of size 100 for different values of level α; the comparisons are made through Kolmogorov-Smirnov tests.

3.1 Binary Markov components

We assume that components can be either working (state 0) or under repair (state 1). The function f is the identity. In this case explicit computations can be completely carried out.

We denote by λ the failure rate and by μ the repair rate. Easy computations lead to the following critical time:

$$\forall 0 < \alpha < \frac{\lambda}{\lambda + \mu}, \quad t_\alpha = -\frac{1}{\lambda + \mu} \log \left(1 - \frac{\alpha(\lambda + \mu)}{\lambda} \right) \ , \tag{7}$$

and the asymptotic variance σ^2 is:

$$\sigma^2 = \frac{\alpha(1-\alpha)}{(\lambda - \alpha(\lambda + \mu))^2} \ . \tag{8}$$

Table 1 contains results of simulation for $\lambda = \mu = 1$.

Table 1: Comparisons for a system with Markovian binary components

α	t_α	σ^2	K.-S. stat.	p-value
0.10	0.112	0.141	0.785	0.568
0.20	0.255	0.444	0.555	0.918
0.30	0.458	1.312	1.203	0.111
0.40	0.805	6.	1.166	0.132

3.2 Example of approximation for general binary components

We now assume that failure and repair distributions are not necessary exponentially. However it is well-known that any distribution with values on $[0, +\infty[$ can be approximated by a phase-type distribution. Thus one can replace the initial distributions by phase-type approximation (with an appropriated choice of state space E and degradation function f) and then apply theorem 1.

In this subsection we treat only the following example: let us consider that failure times are exponentially distributed with parameter λ, and that the repair times are now deterministic and equal to $1/\mu$. The latter distribution can be approximated by the Erlang distribution with parameters $(r, r\mu)$ where r is a positive integer. Hence we have to choose $E = \{0, 1, \ldots, r\}$ and f such that $f(e) = 0$ if $e = 0$ and 1 otherwise.

Notice that we do not actually prove the convergence as r tends to infinity; we just observe on numerical results that the mean degradation of these two systems are close: see table 2 for computations of the maximal difference when $\lambda = \mu = 1$ and for various values of r. In table 3, results of simulations are recorded for $r = 100$ and $\lambda = \mu = 1$.

Table 2: Maximal difference between the exact value of m and a phase-type approximation, for a binary system with deterministic repair times.

r	10	20	30	40	50	60	70	80	90	100
Max. diff.	0.091	0.069	0.059	0.053	0.048	0.044	0.041	0.039	0.037	0.035

Table 3: Comparisons of empirical distributions of hitting times, with asymptotic distributions on a phase type approximation.

α	t_α	σ^2	K.-S. stat.	p-value
0.05	0.051	0.053	0.920	0.366
0.10	0.105	0.111	1.004	0.266
0.15	0.163	0.176	0.829	0.497
0.20	0.223	0.25	0.827	0.501
0.25	0.288	0.333	0.773	0.589
0.30	0.357	0.428	0.792	0.557
0.35	0.431	0.538	1.003	0.267
0.40	0.511	0.666	0.681	0.743
0.45	0.598	0.818	0.957	0.318
0.50	0.693	1.	0.703	0.706

Remark In models 3.1 and 3.2, if we consider non-repairable components (i.e. $\mu = 0$), theorem 1 is just a particular case of a well-know Central Limit Theorem for order statistics. Indeed in that case the failure time of the system is equal to the k-th order statistic of a i.i.d. sequence of random variables.

3.3 A model with multi-state components

Multi-state components can take into account performance levels and progressive degradations. Here we will study a model based on (Pham, Suprasad, and Misra 1996). Three states are considered: 0 (perfect), 1 (deteriorated) and 2 (failed). A perfect component can either deteriorate or directly fail. A deteriorated component can either fail or be repaired. A failed component cannot be repaired. Such components are said to be partially repairable. In order to distinguish between a perfect and a deteriorated component we choose function f as follows: $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(2) = 1$. Even state 2 is absorbing the system cannot be reduced to a system with binary components and a phase-type distribution of failure times. Table 4 contains results of simulations of such a system with all transition rates equal to 1.

Table 4: Comparisons for a system with partially repairable components.

α	t_α	σ^2	K.-S. stat.	p-value
0.10	0.071	0.044	0.772	0.590
0.20	0.154	0.106	0.633	0.818
0.30	0.252	0.196	0.524	0.946
0.40	0.371	0.333	0.929	0.354
0.50	0.517	0.550	0.674	0.755
0.60	0.707	0.914	0.450	0.987
0.70	0.964	1.576	0.942	0.337
0.80	1.344	2.971	0.967	0.306
0.90	2.021	7.203	0.663	0.771

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References

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