An Approach to Reliability Systems Using PH-distributions

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Abstract

Two reliability systems are considered in this work. First, a unit system modeled by a semi-Markov process, and then a two-system governed by a non-Markovian stochastic process. In both cases an approach is followed using phase-type distributions, showing the versatility of this class of distributions in the modeling of reliability systems. Performance measures are determined and presented in an algorithmic form for their computational implementation.

1. Introduction

A repairable unit system subjected to external and internal failures, with operational and repair times following general distributions is considered. There is a repairman, and the failures can be repairable or non-repairable. Different types of repairing are incorporated into the system via geometric processes. After a fixed number of repairs, a new one replaces the system to the next failure. A semi-Markov process governs this model, and some performance measures are calculated. This model incorporates different effects, generalizing previous papers (Ravichandran 1990; Lam 1997; Neuts et al. 2000). A two-system with units as above and two repairmen is also studied. The model that describes the system is a non-Markovian process, and thus the expressions are complex, and the application is not possible in a tractable form. We show that an approach using phase-type distributions (PH distributions) is possible for both systems. Explicit and handle expressions for the performance measures are calculated and the algorithmic expressions allow them to be applied following computational programs. Moreover, in the two-system approach, we incorporate into the model the possibility of remembering the phase of failure of each unit, so the unit returns to the phase of failure after repair.

2. A unit system

We consider a unit system. It can be up when operational, and down when in repair. These times are random. We make the following assumptions:

1. External failures occur according to a Poisson process of rate λ. These can be repairable, with probability p, and non-repairable, with probability $1 - p = q$. Internal failures are non repairable. All failures are independent.
2. There is a repairman. The successive operational times for a geometric process, and the same for the successive repair times.
3. The lifetime U_n of the system after the repair (n–1) follows a distribution function $F_n(x) = F(a^{n-1} x)$, $a > 0$, $x \geq 0$. The repair time D_n after the failure n follows a distribution function $G_n(x) = G(b^{n-1} x)$, $b > 0$, $x \geq 0$. Both have finite mean and are non-null. Sequences $\{U_n, n \geq 1\}$ and $\{D_n, n \geq 1\}$ are independent.
4. The system is replaced by a new and identical one after having undergone N repairs.

The successive operational times are given in terms of the distribution function of the system when it is new, denoted by $F = F_1$, and the same for the repair times, with $G = G_1$. Sequences $\{U_n, n \geq 1\}$ and $\{D_n, n \geq 1\}$ are geometric processes.

The states of the system are divided in two groups: up states and down states. The up states will be denoted by i, the operating state after i repairs, $i = 0, 1,\ldots, N$, and the down state by i_R, the repair state after $i - 1$ repairs, $i_R = 1_R, 2_R,\ldots, N_R$. Under these assumptions, the stochastic process $\{Z(t), t \geq 0\}$ that governs the system is a semi-Markov process (SMP) with the state space given $E = \{0, 1_R, 1, 2_R, 2,\ldots, N_R,, N\}$. It will be assumed that it has right continuous sample functions.

2.1 Semi Markov kernel

Denoting t_n, n = 0, 1,… the transition epochs, the state entered into at time t_n is denoted by $Y_n = Z(t_n)$, so that $\{Y_n, n \geq 0\}$ is the embedded Markov chain of the SMP. The sojourn times in successive states are denoted by $\tau_n = t_{n+1} - t_n$.
The semi-Markov kernel, denoted by $\mathbf{Q}(x) = (Q_{ij}(x))$, $i,j \in E$, $x \geq 0$, is defined by $Q_{ij}(x) = P\{Y_{n+1} = j, \tau_n \leq x | Y_n = i\}$. These quantities can be determined applying the standard calculations in SMP. Given the particular system we are considering, the only non-null entries in the semi-Markov kernel are those corresponding to the transitions $i \to 0$, for $i = 0, 1,\ldots, N$, when a replacement ococurs; the transitions $i_R \to i$, for $i_R = 1,\ldots, N$, when a repair is completed; and the transitions $i \to (i+1)_R$, for $i_R = 0,\ldots, N-1$, when a repairable failure arrives. The authors have calculated the mathematical expressions of these entries.

2.2 Stationary probability vector

From the semi-Markov kernel, it is possible to determine the transition matrix $\mathbf{P} = (p_{ij})$ of the embedded Markov chain. The procedures we apply enable us to calculate the matrix $\mathbf{P}$ in terms of the Laplace Stieltjes transform (LST) ϕ of the distribution function F via the Laplace transform of the entries Q_{ij} of the semi-Markov kernel. Moreover, it is possible to determine the mean times n states η_j, $j \in E$. Once the matrix $\mathbf{P}$ is determined, the stationary probability vector for the embedded Markov chain, denoted by $\boldsymbol{\pi} = (\pi(j), j \in E)$, is calculated from the equation $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}$. Recurrent expressions are formulated from the equations $\pi(i) = \pi(i_R)$, $\pi(i_R) = \pi(i - 1)p_{i-1,iR}$, for $i = 1,\ldots, N$; and from the normalization equation, $\pi(0)$ is determined. It is well known that the stationary probability vector $(p(j)$ $j \in E)$ for the SMP is given by the expression

$$p(j) = \frac{\pi(j)\eta_j}{\sum_{k \in E} \pi(k)\eta_k}, \quad j \in E.$$

It can be shown that this vector depends on the LST ϕ, the mean time of the first repair, and of course of the parameters of the model a, b, p, λ.

2.3 Performance measures

The stationary availability of the system, denoted by A, is $A = p(0) + p(1) + \cdots + p(N)$. The other measures we focus on are the rate of occurrence of the different type of failures in the steady state. The key renewal theorem for semi-Markov processes has been used to formulate explicit expressions for these rates. As expected, all the expressions depended on the LST transform ϕ and the mean time of the first repair.

2.4 Approach for a Markov system

As we have shown in previous sections, the availability and the rates of occurrence of failures depends on the stationary distribution, and this on the LST ϕ and the mean time of the first repair. The application of this model to a physical system will depend on the mathematical form of the function ϕ. For example, if this function is algebraic, the previous formulae may be applicable and the corresponding computational implementation can be performed. This is the case when we take F and G, PH distributions as distribution functions. Thus, all the expressions above are algebraic, and present an algorithmic form; consequently, the applicability of these formulae can be calculated in a natural way.

3. A two-system: approaching for a Markovian system

In this section we consider a two-system with independent units. Each unit is as in Section 2– that is, subjected to the assumptions 1-4 given above. There is a repairman for each unit. The system is operational if both units are. The states of the system are given by couples (i, j), $i = 0, 1_R, 1, 2_R,,2,\ldots, N_{1R}, N_1$; $j = 0, 1_R, 1, 2_R,,2,\ldots, N_{2R}, N_2$; where the component k (= 1, 2) indicates the state of the unit k. It is illustrative to build the transition diagram. The drawback of the stochastic process that results is that it is not Markovian and, consequently, the expressions for the quantities of interest are difficult to calculate. For becoming a Markov system, new assumptions related to exponential times might be considered. We will use F_k and G_k to denote the distribution functions for the operational and repair times, respectively, of the unit k. It is known (Neuts 1981) that the PH distribution class is dense in the class of the non-negative distribution functions. Thus, a Markovian approach to this system can be performed taking F_k, G_k, as PH distributions. In this way, we can approach a general stochastic process by a Markov one. From the assumptions given in Section 2, we will simplify, assuming that each unit can be repaired only one time, $N_1 = N_2 = 1$, so that they are replaced at the second repairable or non-repairable failure. The versatility of the PH distributions will be stand out, under the assumption that when a unit is repaired, it returns to the operational phase in which the failure occurred.
The following notation will be used: the lifetime of unit k when new follows a PH distribution with representation given by $(\cdot, \mathbf{T})$, and $(\cdot, a_k\mathbf{T})$ after repairing, depending on the initial vector of the operational phase in which the failure occurred. This is the case that will be studied in the present work. When there is no memory of this phase, the representations above are given by $(\alpha, \mathbf{T})$ and $(\alpha, a_k\mathbf{T})$, respectively. The repair time of the unit k follow a PH distribution with representation given by $(\beta, \mathbf{S})$. The number of operational and repair phases for the unit k are m_k and n_k.

3.1 Generator

When PH distributions are involved, the states (i, j) of the general stochastic process are considered macro-states of the system, following Neuts et al. (2000). We introduce the following notation, for k = 1, 2: wk: unit k is new and it is operational, wrk: unit k is operational and it has been repaired, and rk: unit k is in repair. Then, the macro-states are classified depending on the operational or in repair stage of the units. Four classes are distinguished: 1) The two units are operational, E_0 = {(w1,w2), (w1,wr2), (wr1,w2), (wr1,wr2)}; 2) only unit 1 is operational, E_1 = {(w1,r2), (wr1,r2)}; 3) only unit 2 is operational, E_2 = {(r1,w2), (r1,wr2)}; and 4) both units are in repair, E_3 = {(r1,r2)}.
The macro-states of E_0 are of order $1 \times m_1m_2$, the ones of E_1 are of order $1 \times m_1n_2m_2$, the ones of E_2 are of order $1 \times m_1n_1m_2$, and the ones of E_3 are of order $1 \times m_1n_1m_2n_2$. The generator $\mathbf{Q}$ can be expressed in terms of the blocks that determine the set E_i. It can be seen that the final form is:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{Q}_{00} & \mathbf{Q}_{01} & \mathbf{Q}_{02} & \mathbf{0} \\ \mathbf{Q}_{10} & \mathbf{Q}_{11} & \mathbf{0} & \mathbf{Q}_{13} \\ \mathbf{Q}_{20} & \mathbf{0} & \mathbf{Q}_{22} & \mathbf{Q}_{23} \\ \mathbf{0} & \mathbf{Q}_{31} & \mathbf{Q}_{32} & \mathbf{Q}_{33} \end{pmatrix}.$$

Each block is determined taking into account the transition between macro-states. The Kronecker product is involved due to the memory of the failure stage of the units.

3.2 Stationary probability vector

We use $\boldsymbol{\pi}_k = [\boldsymbol{\pi}_k(0), \boldsymbol{\pi}_k(1_R), \boldsymbol{\pi}_k(1)]$ to denote the stationary probability vector for each unit (Neuts et al. 2000), and will express the stationary probability vector of the system, denoted by $\boldsymbol{\pi}$, by means of $\boldsymbol{\pi}_k$, k = 1, 2. It is useful to represent it according with the blocks of the matrix **Q**. Ordering the macro-states as {(w1,w2), (w1,wr2), (wr1,w2), (wr1,wr2); (w1,r2), (wr1,r2); (r1,w2), (r1,wr2); (r1,r2)}, the form of the vector $\boldsymbol{\pi}$ is $\boldsymbol{\pi} = [\boldsymbol{\pi}_{01}, \boldsymbol{\pi}_{02}, \boldsymbol{\pi}_{03}, \boldsymbol{\pi}_{04}; \boldsymbol{\pi}_{11}, \boldsymbol{\pi}_{12}; \boldsymbol{\pi}_{21}, \boldsymbol{\pi}_{22}; \boldsymbol{\pi}_{31}]$, where $\boldsymbol{\pi}_{01}$ is the vector of order $1 \times m_1m_2$ corresponding to (w1, w2), and the same for $\boldsymbol{\pi}_{02}$, $\boldsymbol{\pi}_{03}$, $\boldsymbol{\pi}_{04}$ and (w1,wr2), (wr1,w2), (wr1,wr2), properly ordering the phases. The rest of sub-vectors of $\boldsymbol{\pi}$ are built in the same way. The final expression for the components of $\boldsymbol{\pi}$ can be determined by an algorithmic procedure, which is illustrated with the first component. If $\mathbf{A}^T$ denotes the transpose of matrix or vector **A**, for sake of the independence of the units, the sub-vector $\boldsymbol{\pi}_{01}$ is determined calculating the product of vectors $[\boldsymbol{\pi}_1(0)]^T[\boldsymbol{\pi}_2(0)]$ found for each unit. That is, it is a matrix of dimension $m_1 \times m_2$; expressing this matrix as a column vector of m_2 blocks of row vectors each one of order $1\times m_1$, $([\boldsymbol{\pi}_{12}(0)_1], [\boldsymbol{\pi}_{12}(0)_2], \cdots, [\boldsymbol{\pi}_{12}(0)_2)])^T$, and then $\boldsymbol{\pi}_{01}$ is the vector of order $1 \times m_1m_2$ given by $([\boldsymbol{\pi}_{12}(0)_1], [\boldsymbol{\pi}_{12}(0)_2], \cdots, [\boldsymbol{\pi}_{12}(0)_2)])$. The others blocks are formulated in a similar way. This algorithmic procedure prepares the computational implementation.

3.3 Performance measures

For a general two-system under the assumptions of the present paper, the performance measures can be calculated in terms of the stationary probability vector, as in the unit system. For example, in the model above, the non-operational macro-state is E_3. Thus, the stationary availability A is the probability that the system does not occupy E_3, and it is $A = 1 - \boldsymbol{\pi}_{31}\mathbf{e}$, **e** being a column vector of 1´s of the order $m_1n_1 \times m_2n_2$. This expression can be given in terms of the components of the vector $\boldsymbol{\pi}$. For example, the rate of occurrence of repairable failures of the system is $\nu_1 = \lambda p(\boldsymbol{\pi}_{11}\mathbf{e} + \boldsymbol{\pi}_{21}\mathbf{e})$, the vectors **e** being of the orders $1 \times m_1m_2n_2$ and $1 \times m_1n_1m_2$, respectively. For the external non-repairable failures the rate of occurrence of failures is

$$\nu_2 = (\boldsymbol{\pi}_{11}, \boldsymbol{\pi}_{12})\begin{pmatrix} \lambda q\mathbf{e}_{m_1m_2n_2} \\ \lambda\mathbf{e}_{m_1m_2n_2} \end{pmatrix} + (\boldsymbol{\pi}_{21}, \boldsymbol{\pi}_{22})\begin{pmatrix} \lambda q\mathbf{e}_{m_1n_1m_2} \\ \lambda\mathbf{e}_{m_1n_1m_2} \end{pmatrix},$$

and, for internal failures, the vectors $\boldsymbol{\pi}_{11}$, $\boldsymbol{\pi}_{12}$, $\boldsymbol{\pi}_{21}$, $\boldsymbol{\pi}_{22}$ are involved, (these indicating that only one unit is operational) in addition to the absorption vectors of the representation of the PH distributions.

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