Linking Dominations and Signatures in Network Reliability

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Abstract

Comparing the reliability of two networks of more than modest size can be a computationally intensive exercise. In this paper, domination theory and the notion of the signature of a network, and their respective roles in calculating the reliability of a network, are briefly reviewed. The computational advantages of the former, and the interpretive richness of the latter, beg the question: how are the two related? The exact functional relationship between the signature vector and the vector of signed dominations is obtained. A detailed example is given in which the connection between these two concepts is usefully exploited.

1 Introduction

A network G with v vertices and n edges is typically denoted by the symbol $G(v, n)$. In communication networks, as in many other types of networks, the primary quality characteristic of interest is connectivity. For example, a two-terminal network is connected if there is at least one set of functioning edges providing a path from one terminal to the other.

The network characteristics upon which we will be focusing are defined in terms of edges whose lifetimes are treated as independent and identically distributed random variables. We will be concerned with the distribution of T, the failure time of the network and, in particular, with the probability that it is connected at a given time t_0. Here, we'll treat the states of edges at time t_0 as independent Bernoulli variables, and will restrict attention to two-terminal reliability.

It is known, of course, that the reliability of the network $G(v, n)$ in i.i.d. edges can be expressed as a polynomial $h(p)$ of order n, that is, as

$$h(p) = \sum_{r=1}^{n} d_r p^r, \tag{1}$$

where p is the common success probability for the edges. Satyarananaya and Prabhakar (1978) showed that the coefficients in (1) could be obtained as the signed dominations associated with the network. Indeed, domination theory, described as a breakthrough by Agrawal and Barlow (1984) among computational tools in network reliability, continues to be a widely used algorithmic vehicle for calculating the reliability polynomial. We review the concept of dominations in Section 2. We note that, as useful as domination theory has proven to be in simplifying the computation of the reliability of a network, it has not been found particularly useful in comparing one network design with another as is required, for example, in searching for universally optimal networks of a given size (v, n).

A quite different tool was introduced by Samaniego (1985) for studying the performance properties of coherent systems. The concept of signature applies equally well to network reliability. The signature of a network is a probability vector $\boldsymbol{s}$ whose components are simply the respective probabilities that the first, second, $\cdots$, and nth edge failures (ordered by time of occurrence) are fatal to the network. Unlike the domination vector, the properties of the signature vector are readily interpretable and have a close relationship to the failure time T of the network itself. We review the notion of signature, and some of the problems to which it has been applied, in Section 3.

The main goal of this paper is to identify the exact relationship between the vector of signed dominations $\boldsymbol{d}$ and the signature vector $\boldsymbol{s}$. This is accomplished in Section 4. Our closing example demonstrates the utility of this linkage.

2 A Brief Look at Domination Theory

The notion of dominations was discovered in the process of seeking a reduction in the complexity of the well-known inclusion-exclusion formula for calculating the probability that all edges are functioning in at least one of a given network's minimal path sets. The inclusion-exclusion rule applies to the union of any m sets, and may be written as

$$\begin{aligned} P\left(\bigcup_{i=1}^{m} A_i\right) &= \sum_1 P(A_i) - \sum_2 P(A_i \cap A_j) + \sum_3 P(A_i \cap A_j \cap A_k) \\ &- \cdots + (-1)^{m+1} P\left(\bigcap_{i=1}^{m} A_i\right), \end{aligned} \tag{2}$$

where $\sum_i$ represents a sum over all i-fold intersections. If A_i in (2) represents the event that all edges in the ith minimal path set are working, and there are m minimal path sets in all, then the formula in (2) provides the probability that the network will function.

Suppose that one has a list of minimal path sets of a given network in n i.i.d. edges. A *formation* is defined as a union of minimal path sets. A formation is thus the union of the edges in a fixed collection of minimal path sets. An *i-formation* is a union of the components in a set of i minimal path sets. For example, the union $\{1, 2, 3, 4\}$ of the minimal path sets $\{1, 2\}$, $\{2, 3\}$, and $\{3, 4\}$ would be an example of a formation that is both a 2-formation and a 3-formation. We will refer to a particular formation as even if it is the union of an even number of minimal path sets and as odd if it is the union of an odd number of minimal path sets. Clearly, a given formation can be both odd and even. The *signed domination* of a given union of minimal path sets is simply the difference between the number of even dominations and the number of odd dominations for that union. Satyarananaya and his co-workers showed that is a large variety of network reliability settings, the awkward expression for network reliability in (2) could be replaced by the simple form of the reliability polynomial in (1), where $(d_1...d_n)$ is the vector of signed dominations.

3 A Brief Look At Signatures

The signature of a network of order n (that is, having n edges) is defined as the probability distribution $\boldsymbol{s}$ on the integers $\{1, 2, \cdots, n\}$ for which

$$s_i = P(T = X_{(i)}), \quad i = 1, 2, \cdots, n, \tag{3}$$

where $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ are the order statistics from a random (i.i.d.) sample drawn from the (arbitrary) continuous lifetime distribution F, and T is the lifetime of the network.

The fact that the signature $\boldsymbol{s}$ depends only on the network design, and not on the distribution F, is a consequence of the fact that each of the $n!$ orderings of the failure times $X_1, X_2, \cdots, X_n$ of the n edges is equally likely to occur under the i.i.d. assumption. Thus, the probability that the ith edge failure is fatal to the network is solely dependent on the likelihood that the last working edge in some minimal cut set is the ith edge to fail overall.

As shown in Samaniego (1985) (see also Kochar, Mukerjee and Samaniego (1999)), the survival function of a network's lifetime T can be written as a simple function of $\boldsymbol{s}$ and F. When focusing on the reliability of the network at a fixed time t_0, where $P(X_j > t_0) = p$ for all j, this representation reduces to the reliability polynomial in pq-form, that is, in the form

$$h(p) = \sum_{j=1}^{n} \left(\sum_{i=n-j+1}^{n} s_i \right) \binom{n}{j} p^j q^{n-j}. \tag{4}$$

If a_j is the proportion of path sets among the $\binom{n}{j}$ sets of j working components (with other components non-working), then we see that the reliability polynomial can be written as

$$h(p) = \sum_{j=1}^{n} a_j \binom{n}{j} p^j q^{n-j}. \tag{5}$$

It follows that $\boldsymbol{a}$ and $\boldsymbol{s}$ are linearly related. For future reference, the linear relationship between the vectors $\boldsymbol{a}$ and $\boldsymbol{s}$ is denoted as $\boldsymbol{a} = \boldsymbol{P}\boldsymbol{s}$.

In the introduction, we alluded to the fact that signatures are rich in interpretation and are particularly useful in the comparison of competing networks. We summarize here a collection of results that support this remark. The random variables X_1 and X_2, discrete or continuous, are stochastically ordered (i.e., $X_1 \leq_{st} X_2$) if the survival functions $S_i(x) = P(X_i > x)$ are ordered, that is, if $S_1(x) \leq S_2(x)$ for all x. We say that X_1 is smaller than X_2 in the hazard rate ordering ($X_1 \leq_{hr} X_2$) if the ratio of survival functions $S_2(x)/S_1(x)$ is nondecreasing in x. Finally, X_1 is said to be smaller than X_2 in the likelihood ratio ordering ($X_1 \leq_{lr} X_2$) if the ratio $f_2(x)/f_1(x)$ is nondecreasing in x, where f_i represents the density or probability mass function of X_i. As is well known, $lr \Rightarrow hr \Rightarrow st$. With this notation, we may now restate results from Kochar, Mukerjee and Samaniego (1999) relating properties of signatures to properties of network lifetimes.

Theorem 1 *Let $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ be the signatures of two networks with n i.i.d. edges, and let T_1 and T_2 be the corresponding lifetimes of these networks. If $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$ or $\boldsymbol{s}_1 \leq_{hr} \boldsymbol{s}_2$ or $\boldsymbol{s}_1 \leq_{lr} \boldsymbol{s}_2$, then $T_1 \leq_{st} T_2$ or $T_1 \leq_{hr} T_2$ or $T_1 \leq_{lr} T_2$, respectively.*

4 The Linkage Between Domination And Signatures

The linear relationship between the vectors $\boldsymbol{d}$ and $\boldsymbol{a}$ is apparent by comparing equations (1) and (5). Let us denote that relationship by $\boldsymbol{d} = \boldsymbol{M}\boldsymbol{a}$. Our interest is in the expression for $\boldsymbol{s}$ in terms of $\boldsymbol{d}$, that is, in $\boldsymbol{s} = \boldsymbol{P}^{-1}\boldsymbol{M}^{-1}\boldsymbol{d}$. Our main result is as follows:

Theorem 2 *Let $\boldsymbol{d}$ and $\boldsymbol{s}$ denote the domination and signature vectors for a given network of order n. Then for $i = 1, \cdots, n$ we have*

$$s_i = \sum_{j=1}^{n-i} \frac{-(n-i)_j + (n-i+1)_j}{(n)_j} d_j + \frac{(n-i+1)_{n-i+1}}{(n)_{n-i+1}} d_{n-i+1}, \tag{6}$$

where $(k)_j$ denotes the number of ways of selecting without replacement j items from k items, accounting for the order of selection. For a proof of this result, see Boland, Samaniego and Vestrup (2001).

For two complex networks, the difference polynomial $\sum (d_{2r} - d_{1r}) p^r$ will typically be of quite high degree. Thus, determining whether one reliability polynomial is uniformly larger than another for all $0 < p < 1$ is a task equivalent to finding the roots of a high degree polynomial. Thus, closed-form expressions for the solutions of such problems are not possible in general.

As an example, consider the comparison between the two networks of order G(9,27), pictured below (Figure 1). In this case, the difference polynomial referred to above is of degree 19. While the uniform superiority of one of these networks over the other is certainly not obvious by inspection, one could by numerical means show that $h_{AX3}(p) \geq h_{AX4}(p)$ for all $p \in [0, 1]$. But the comparison of the two

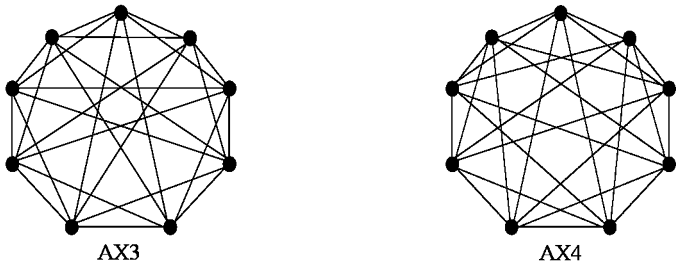

Figure 1:

signatures immediately yields a stronger conclusion. We will demonstrate, using the signatures of these two networks, that AX3 is indeed uniformly superior to AX4, in both the stochastic ordering sense ($h_{AX3}(p_1) \geq h_{AX4}(p)$) as well as in the stronger hazard rate ordering sense.

The main thesis of this note can be summarized quite succinctly: Domination theory is a useful tool in making network reliability calculations. However, for the purpose of comparing the performance characteristics of two competing networks, reliabilities expressed in terms of signed dominations will tend to have little intuitive content and may be of limited use (except for the possibility of brute force computation). The utility of signatures in the comparison of networks of the same size immediately raises questions about the exact relationship between the domination and signature vectors. The functional relationship linking the signature vector with the vector of dominations is displayed above. This linkage allows one to combine the computational advantages of domination theory with the intuitive and interpretive qualities of signatures for the purpose of making comparisons among networks. The growing but still inconclusive literature on the existence, uniqueness and identification of uniformly optimal networks of a given size (see for example, Ath and Sobel (2000), Boesch et al. (1991), Myrvold et al. (1991), and Wang (1994)) should benefit from the application of these linked tools.

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