Exact and Approximated Reliability of a 2-dimensional consecutive *k*-out-of-*n*:F system

Tetsushi Yuge
Dept of Electrical and Electronic Engineering
National Defense Academy
239-8686 Yokosuka
Japan
yuge@nda.ac.jp

Masaharu Dehare

&

Shigeru Yanagi

Abstract

An approximation method to obtain the reliability of a 2-dimensional consecutive *k*-out-of-*n*: F system is discussed. Although analysis to obtain exact reliability requires many calculation resources for a system with a large number of components, the proposed approximation method obtains the reliability easily by giving an assumption on the maximum number of failed components in an operable system. This approximated reliability is exact when the total number of failed components is less than the assumed maximum number. The accuracy of the new method is confirmed by numerical examples.

1. Introduction

A 2-dimensional consecutive *k*-out-of-*n*:F system ($(k,k)/(n,n)$ system in this paper) is literally a 2-dimensional version of a consecutive-*k*-out-of-*n*:F system which has been extensively studied. This system consists of n^2 components arranged like an (n,n) matrix and fails if and only if the system has an (k,k) submatrix that contains failed components only. A consecutive-(r,s)-out-of-(m,n):F system is a generalized version of a $(k,k)/(n,n)$ system. These systems have been introduced by *Salvia and Lasher (1990)* and can be applied to obtain the reliability of a sensor system, an X-ray diagnostic system, a pattern search system, a liquid crystal display system, a phased array radar system and so on.

The recursive equation approach has been studied to obtain the exact reliability. Although these methods are effective for a relatively small system, the methods need more computation time as the system size becomes larger. Therefore, for practical systems mentioned above, another effective method is required for reliability analysis. The exact reliability is obtained for a 1-dimensional system using a combinatorial approach. However this method has not been applied so far for a 2-dimentional version because of its complexity.

In this paper, we propose a new approach for a conditional $(k,k)/(n,n)$ system which has a condition on the number of allowable failed components in the system. A state of the system having more failed components than the number of allowable failed components is assumed to be failed. The reliability is calculated directly by using a combinatorial equation that does not depend on the system size. The maximum number of failed components is given beforehand. The number is assumed to be $2k^2$ in this paper. For a system without this assumption the approximated reliability is obtained. In this method, the different kind of cut sets in each number of failed components are enumerated in order to obtain exact system reliability. The procedure of this method is explained by using the case of k=2 for simplicity.

Abbreviations & notations

$\{\tilde{C}_i\}$ **:** set of minimal cut set

MMC : Multiple minimal cut set ($\tilde{C}_i \cup \tilde{C}_j \cup \ldots;\ \tilde{C}_i \cap \tilde{C}_j \cap \ldots \neq f$)

PMC : product of minimal cut sets ($\tilde{C}_i \cap \tilde{C}_j \cap \ldots$)

K, N : k^2, n^2
p, q : probability that a component is in a normal(failed) state
c_m : $=\{c_{m,j}\}$, where $c_{m,j}$ is the jth cut set having m failed components
w_m : $=\{w_j^{(m)}\}$, where $w_j^{(m)}$ is the jth cut set which includes more than or equal to two different minimal cut sets and m failed components
$a_{m,s}$: $=\{a_j^{(m,s)}\}$, where $a_j^{(m,s)}$ is the jth cut set such that $\tilde{C}_k, \tilde{C}_l, \ldots \subset w_i^{(m)}, \left|\tilde{C}_k \cap \tilde{C}_l \cap \ldots\right| = s$.
$|\cdot|$: the cardinal number of a set
a conditional (k,k)/(n,n) system : The assumptions for a conditional (k,k)/(n,n) system is as follows.

(1) System consists of N identical components arranged like an (n,n) matrix.
(2) Both component and system are either normal or failed.
(3) The failure times of the components are identically and independently distributed.
(4) The system fails if (1) any (k,k) sub matrix fills with failed components only or (2)there are more than or equal to $2K$ failed components in the system.

2. Reliability of a conditional (*k,k*)/(*n,n*) system

The states of a conditional (k,k)/(n,n) system are classified into three groups subject to the number of failed components, m.

1. $m \le K-1$ normal
2. $K \le m \le 2K-1$ normal or system down
3. $m \ge 2K$ system down

Then the reliability is given as follows;

$$R = \sum_{m=0}^{2K-1} \binom{N}{m} p^{N-m} q^m - \sum_{m=K}^{2K-1} |c_m| p^{N-m} q^m . \tag{1}$$

Since $|c_m|$ is the number of different cut sets having m failed components in the system, the problem of obtaining a reliability results in the problem of obtaining $|c_m|$ for each m.

2.1 $|c_m|$ for a conditional (2,2)/(*n,n*) system

(a) $4 \le m \le 5$

If $4 \le m \le 5$, $c_{m,j}$ consists of one minimal cut set, $\tilde{C}_l$, and m-4 failed components which are not in $\tilde{C}_l$. Since the total number of minimal cut sets in an (n,n) lattice system is $(n-k+1)^2 = (n-1)^2$. $|c_m|$ is given by the following equation.

$$|c_m| = (n-1)^2 \binom{N-4}{m-4}, \quad (m=4,5). \tag{2}$$

(b) $m \ge 6$

$|c_m|$ is obtained in the same way as the case of $4 \le m \le 5$. Namely, it is given by enumerating the combinations of a minimal cut set and failed components which do not belong to the minimal cut set. However for $m \ge 6$ there is a system state that two minimal cut sets exist in the system(Fig.1). Then $|c_m|$ is given as follows.

$$|c_m| = (n-1)^2 \binom{N-4}{m-4} - |w_m| \tag{3}$$

(Note that any cut set having m failed components include at most two different minimal cut sets if m<8)

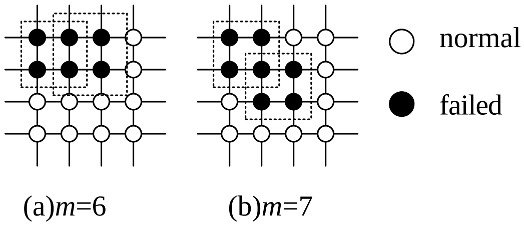

Figure 1: Examples of a cut set

Table 1: Possible $a_{m,s}$ for each m

m	$a_{m,s}$
4, 5	-
6	$a_{6,2}$
7	$a_{7,1}$, $a_{7,2}$

The broken lines show minimal cut sets in Fig.1. In this example, a minimal cut set is not exclusive each other. $|w_m|$ can be obtained by enumerating MMC with m failed components. The MMC in Fig.1(a) belongs to $a_{6,2}$. That of Fig.1(b) belongs to $a_{7,1}$. Table 1 shows possible $a_{m,s}$ for each m. The MMC in Fig.1(a) can be represented that the PMC forms 2×1 sub lattice. So let denote the subset of $a_{6,2}$ whose elements have PMC of 2×1 sub lattice as $a_{6,2\times1}$. $|a_{m,2\times1}|$ can be obtained by giving attention to the smallest sub lattice which covers the MMC in $a_{m,2\times1}$. When the MMC is shown in Fig.1(a), the smallest sub lattice is a (2,3) sub lattice. And the number of (2,3) sub lattices in the system equals to (n-1)(n-2). Therefore $|a_{6,2\times1}|$ is (n-1)(n-2). This is same in case of $a_{6,1\times2}$. Then $|a_{6,2}|$ equals to 2(n-1)(n-2). $|w_6|$ is given as follows.

$$|w_6| = |a_{6,2}| = 2(n-1)(n-2) \tag{4}$$

Therefore, the number of different kind of cut sets, $|c_6|$, is enumerated from eq.(3) as follows.

$$|c_6| = (n-1)^2\binom{N-4}{2} - 2(n-1)(n-2)\,. \tag{5}$$

When m=7, two kinds of $a_{m,s}$ have to be considered. Namely $a_{7,1}$ and $a_{7,2}$. Since these sets are exclusive each other, $|w_7|$ is given as follows.

$$|w_7| = |a_{7,2}| + |a_{7,1}| \tag{6}$$

In the case of $a_{7,1}$ shown in Fig.1(b), the smallest sub lattice which covers the MMC in $a_{7,1}$ is a (3,3) sub lattice. Each (3,3) sub lattice has two MMCs in $a_{7,1}$, and the number of (3,3) sub lattices in the system is $(n-2)^2$, then $|a_{7,1}|$ equals to $2(n-2)^2$. $|a_{7,2}|$ is given as a combination of one MMC in $a_{6,2}$ and one failed component which do not belong to the MMC. From eq.(6), eq.(7) is obtained.

$$|w_7| = |a_{6,2}| \times (N-6) + |a_{7,1}| = 2(n-1)(n-2)(N-6) + 2(n-2)^2 \tag{7}$$

Therefore $|c_7|$ is given by using eqs.(3,7) as eq.(8).

$$|c_7| = (n-1)^2\binom{N-4}{3} - 2(n-1)(n-2)(N-6) - 2(n-2)^2 \tag{8}$$

Finally, $|c_m|$ is enumerated when k=2 as follows, where $\binom{x}{y} \equiv 0$ for y<0.

$$|c_m| = (n-1)^2\binom{N-4}{m-4} - 2(n-1)(n-2)\binom{N-6}{m-6} - 2(n-2)^2\binom{N-7}{m-7},\ (4 \le m \le 7). \tag{9}$$

The reliability of a conditional (2,2)/(n,n) system is calculated by substituting eq.(9) to eq.(1).

In general, if PMC forms (a,b) sub lattice, the smallest sub lattice which covers the element of $a_{m,s}$ is a (2k-a,2k-b) sub lattice. The number of (2k-a,2k-b) sub lattices in the system is (n-2k+a+1)(n-2k+b+1).

2.2 $|c_m|$ for *k*=3,4

In case of $k \geq 3$, $|c_m|$ can be enumerated in the same way as the case of *k*=2. When *k*=2, the number of minimal cut sets in a MMC is at most two. However, when $k \geq 3$, more than two minimal cut sets might be included in a MMC. Although the enumeration becomes more complex than that of *k*=2, $|c_m|$ is obtained by calculating the number of combinations of a MMC and failed components which are not in the MMC. $|c_m|$ is obtained when *k*=3 as follows.

$$|c_m| = (n-2)^2\binom{N-9}{m-9} - 2(n-2)(n-3)\binom{N-12}{m-12} - 2(n-3)^2\binom{N-16}{m-14} - 4(n-3)(n-4)\binom{N-18}{m-16}$$
$$-2(n-4)\binom{N-17}{m-17} + (n-3)^2\binom{N-16}{m-16}, \qquad (9 \leq m \leq 17).$$

And if *k*=4, $|c_m|$ is given as follows.

$$|c_m| = (n-3)^2\binom{N-16}{m-16} - 2(n-3)(n-4)\binom{N-20}{m-20} - 2(n-4)^2\binom{N-25}{m-23} - 4(n-4)(n-5)\binom{N-28}{m-26}$$
$$-2(n-5)^2\left[\binom{N-36}{m-28} + 8\binom{N-36}{m-29} + 23\binom{N-36}{m-30} + 32\binom{N-36}{m-31}\right] - 4(n-4)(n-6)\binom{N-31}{m-29}$$
$$-4(n-5)(n-6)\binom{N-30}{m-30} - 2(n-6)^2\binom{N-31}{m-31} + (n-4)^2\binom{N-25}{m-25}, \qquad (16 \leq m \leq 31).$$

3. Numerical example

Let's confirm the accuracy of reliability approximation when the proposed method is applied to obtain the reliability of a usual (*k*,*k*)/(*n*,*n*) system which has not the restriction of the number of failed components. Fig.2 shows the unreliability of a (2,2)/(10,10) system against the component unreliability, *q*. The proposed method gives good reliability approximation for a system with high component reliability.

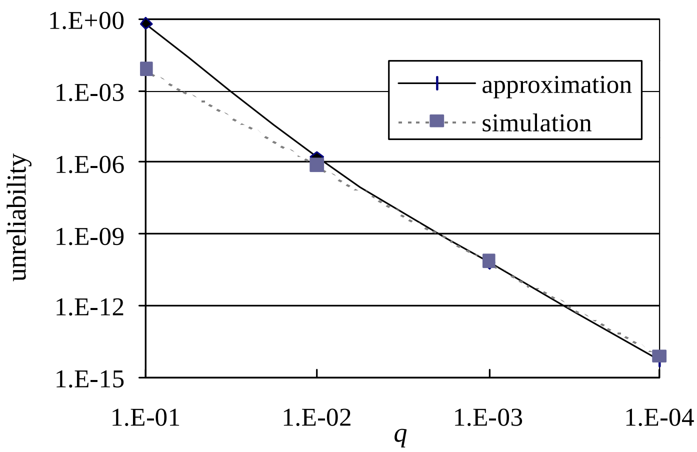

Figure 2: System unreliability of a (2,2)/(10,10) system

References

Salvia, A., Lasher, W. (1990). 2-dimensional consecutive-*k*-out-of-*n*:F models. *IEEE Transactions on Reliability 39* (3), 382-385.