Copulæ and residual lifetimes in time transformed exponential models

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Abstract

We consider bivariate exchangeable lifetimes, and we take into account residual lifetimes. We see how their dependence and, often, their multivariate aging, can be described in terms of copulæ. We study the evolution of dependence as time elapses, especially in the case of time transformed exponential models, where the associated copulæ are Archimedean.

1 Introduction

We[1] consider here two exchangeable lifetimes and we study how their dependence and their aging features evolve as time elapses and they continue to be functioning (alive). In other words, we are interested in the analysis of exchangeable residual lifetimes.

We deal with three main features: dependence, marginal aging, i.e. the one-dimensional aging properties of the marginal laws, and what we may call multivariate aging. The latter notion can be given several interpretations. We consider here multivariate aging in the sense studied by Bassan and Spizzichino (1999, 2000, 2001), which can be roughly summarized as follows: a joint law has positive multivariate aging if, conditionally on a same history of survivals and failures, the marginal law of the residual lifetime of a younger item dominates in some stochastic order sense the corresponding law of an older item.

In Bassan and Spizzichino (2001) it was pointed out that multivariate aging properties of a joint law can be translated into dependence properties of a function B which has all the properties of a copula, except possibly for the fact that it need not be 2-increasing. Recall that a copula, in the simplest case, is a joint distribution function on the unit square with uniform marginals. For an overview of copulæ and of dependence concepts such as TP$_2$, PQD etc., see e.g. Joe (1997) and Nelsen (1999). We refer instead to Marshall and Olkin (1979) for the definition and properties of Schur-concavity and other related notions which can be useful in describing aging (see also Barlow and Mendel (1993) and Spizzichino (2001)). Finally, we refer to Shaked and Shantikumar (1994) for stochastic orders.

Given two exchangeable lifetimes X, Y, we consider their joint survival function $\bar{F}(x,y) = \mathbb{P}(X > x, Y > y)$, their marginal survival function $\bar{G}(x) = \mathbb{P}(X > x) = \bar{F}(x,0)$, the *survival copula*

$$K(u,v) = \bar{F}\left(\bar{G}^{-1}(u), \bar{G}^{-1}(v)\right), \qquad 0 \leq u, v \leq 1,$$

and the *multivariate aging function*

$$B(u,v) = \exp\left\{-\bar{G}^{-1}\left(\bar{F}(-\log u, -\log v)\right)\right\}, \qquad 0 \leq u, v \leq 1.$$

Recall that the survival copula describes the dependence structure of the joint law, and that each of the pairs $\left(B, \bar{G}\right), \left(K, \bar{G}\right)$ completely specifies $\bar{F}$, as the following relations show:

$$\bar{F}(x,y) = K\left(\bar{G}(x), \bar{G}(y)\right) = \bar{G}\left(-\log B\left(e^{-x}, e^{-y}\right)\right).$$

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[1] This note is based on joint work with Fabio Spizzichino

We consider also the joint survival function of the residual lifetimes at epoch r, namely

$$\bar{F}_r\left(x,y\right)=\mathbb{P}\left(X\geq x+r,Y\geq y+r|X\geq r,Y\geq r\right)=\frac{\bar{F}\left(x+r,y+r\right)}{\bar{F}\left(r,r\right)}.$$

Let $K_r,B_r,\bar{G}_r$ denote the survival copula, multivariate aging function and marginal survival function of $\bar{F}_r$, respectively.

The issues we want to address in this extended abstract, and that will be dealt with more extensively in a forthcoming paper, can be summarized in the following list:

- Dependence properties of $\bar{F}$ are reflected in dependence properties of $\bar{F}_r$.
- Schur-concavity properties of $\bar{F}$ can be characterized with properties of the family $\{B_r|r\geq 0\}$.
- Marginal aging properties of $\bar{F}$ are reflected in aging properties of the family $\left\{\bar{G}_r|r\geq 0\right\}$.
- Dependence and Schur-concavity properties of $\bar{F}$ imply aging properties of the family $\left\{\bar{G}_r|r\geq 0\right\}$. Similarly, Schur-concavity properties of $\bar{F}$ and aging properties of $\left\{\bar{G}_r|r\geq 0\right\}$ imply dependence properties of $\left\{\bar{F}_r|r\geq 0\right\}$.
- The family $\{B_r|r\geq 0\}$ depends on B only, whereas $\{K_r|r\geq 0\}$ depends on K and on B.
- K_{r+a} can be more concordant, as concordant, or less concordant than K_r. In other words, dependence may increase or decrease with surviving.

We shall also see how some of the notions above read in the special case of time transformed exponential models.

2 Dependence and aging for residual lifetimes

We give now some results on the issues described above. All the proofs will be given in a forthcoming paper. First, we see how strong dependence properties of $\bar{F}$ imply (weaker) dependence of each $\bar{F}_r$.

Proposition 1. *Let $\bar{F}$ be TP$_2$. Then, for every $r\geq 0$, $\bar{F}_r$ is PQD. Similarly, if $\bar{F}$ is RR$_2$, $\bar{F}_r$ is NQD, for every $r\geq 0$.*

We may now draw a parallel between dependence and aging. The notion of TP$_2$ is stronger than PQD. Similarly, the notion of Schur-concavity, which corresponds to multivariate IFR, is stronger than the property of positive quadrant dependence for the aging function B, which corresponds to multivariate NBU. However, the following characterization holds. It can be summarized as follows: A law is multivariate IFR iff all its residual lifetimes are multivariate NBU.

Proposition 2. *$\bar{F}$ is Schur-concave (respectively: Schur-convex) if and only if $B_r(u,v)\geq uv$ (respectively: $B_r(u,v)\leq uv$) for every $r\geq 0$ and $0\leq u,v\leq 1$.*

The following proposition deals with marginal aging. Again, a stronger notion for $\bar{F}$ is equivalent to a weaker notion for all the laws of the residual lifetimes.

Proposition 3. *The application $x\mapsto\bar{F}(x,y)$ is log-concave (resp: log-convex) for every $y\geq 0$ if and only if $\bar{G}_r$ is NBU (resp: NWU) for every $r\geq 0$.*

The above proposition should be carefully compared with the univariate result stating that X is IFR (hence with log-concave survival function) if and only if the laws $\mathcal{L}\left(X-r|X>r\right)$ are NBU, for every $r\geq 0$.

We turn now our attention to the relations among dependence, Schur-concavity and marginal aging.

Proposition 4. *1. If $\bar{F}$ is Schur-concave and RR_2, then $\bar{G}_r$ is NBU, for every $r \geq 0$.*

2. If $\bar{F}$ is Schur-convex and TP_2, then $\bar{G}_r$ is NWU, for every $r \geq 0$.

3. If $\bar{F}$ is Schur-concave and $\bar{G}_r$ is NWU, for every $r \geq 0$, then $\bar{F}_r$ is PQD, for every $r \geq 0$.

4. If $\bar{F}$ is Schur-convex and $\bar{G}_r$ is NBU, for every $r \geq 0$, then $\bar{F}_r$ is NQD, for every $r \geq 0$.

Notice that Schur-concavity, which is a positive multivariate aging notion, is sufficient for the property of multivariate-NBU of all the residual lifetimes, but not for marginal NBU. It should be observed that in order to obtain a sufficient condition for marginal NBU, one needs Schur-concavity and a strong negative dependence property.

Next, we see what influences the aging functions and the survival copulæ of the residual lifetimes.

Proposition 5. *1. The family $\{B_r | r \geq 0\}$ is determined only by B.*

2. The family $\{K_r | r \geq 0\}$ is determined by K and by B.

Thus, we see that there is no perfect symmetry between multivariate aging and dependence.

3 Residual lifetimes for time transformed exponentials

An exchangeable bivariate law is called a time transformed exponential, and denoted by TTE(W, R), if there exists a strictly decreasing convex survival function W and a strictly increasing function $R : \mathbb{R}_+ \to \mathbb{R}_+$, with $R(0) = 0$ and $\lim_{t\to\infty} R(t) = \infty$ such that the joint survival function can be written in the form

$$\bar{F}(x, y) = W(R(x) + R(y)).$$

The class of TTE models includes several relevant special cases, such as the independent laws, the Schur-constant laws and the case of conditionally i.i.d. r.v.'s with proportional hazards.

It turns out that the survival copula is an Archimedean copula with generator W^{-1}. Recall that a copula is Archimedean with generator ϕ if it can be written in the form

$$\phi^{-1}(\phi(u) + \phi(v)). \tag{1}$$

where $\phi : [0, 1] \to [0, \infty]$ is convex and decreasing, with $\phi(1) = 0$ and, in general, $\phi(0) = \infty$.

The multivariate aging function B is also of the form (1) with $\phi = R \circ (-\log)$, but it need not be a copula, since convexity may fail to hold.

Thus, we see that in TTE(W, R) models, dependence is modulated by W, whereas multivariate aging is governed by R. Marginal aging is regulated both by W and R, since the marginal survival function is $\bar{G} = W \circ R$.

A relation between aging and Archimedean copulæ was studied, in a different context, in Averous and Dortet-Bernadet (2000).

A "nice" feature of TTE models is that the residual lifetimes are again TTE:

Proposition 6. *Let $\bar{F}$ be $TTE(W, R)$. Then $\bar{F}_r$ is $TTE(W_r, R_r)$, with*

$$W_r(x) = \frac{W(2R(r) + x)}{W(2R(r))}, \qquad R_r(x) = R(r + x) - R(r).$$

Thus, the survival copula K_r of $\bar{F}_r$ is again an Archimedean copula, with generator

$$W_r^{-1}(u) = W^{-1}(uW(2R(r))) - 2R(r).$$

This result allows us to draw from the existing literature about concordance ordering and limiting behavior for Archimedean copulæ. In particular, we obtain the following result.

Proposition 7. *Let* $\bar{F}$ *be* $TTE(W,R)$. *Then* K_{r+a} *is less concordant than* K_r *if and only if*

$$W_{r+a}^{-1} \circ W_r(x) = W^{-1}\left(\frac{W\left(2R\left(r\right)+x\right)}{W\left(2R\left(r\right)\right)}W\left(2R\left(r+a\right)\right)\right) - 2R(r+a)$$

is subadditive.

Although, as we have just seen, the survival copula of the residual lifetimes depends on R as well as on W, the fact that concordance decreases monotonically with time does not depend on R. The following proposition deals with this issue.

Proposition 8. *Let* $\bar{F} \sim TTE\left(W,R\right)$ *be such that* $K_{t_2} \leq K_{t_1} \quad \forall t_1 \leq t_2$, *i.e. such that the joint law of the residual lifetimes is less and less concordant as time elapses. Then the same is true for all TTE laws with the same* W *and arbitrary* $\tilde{R}$

Finally, we give some examples in which the concordance of the survival copulæ of the residual lifetimes increases, decreases or stays unchanged.

Example 9. Let $\bar{F}(x,y) = 2(1+e^{x+y})^{-1}$. This law is $TTE(W,R)$, with $W(x) = 2\left(1+e^{x}\right)^{-1}$ and $R(x) = x$. One may check that for every $r, a > 0$, $K_{r+a} \geq K_r$. All these copulæ are NQD, and the limiting copula is the independent copula.

Example 10. Let $\bar{F}$ be $TTE(W,R)$, with $W(x) = \exp\left\{-\sqrt{x}\right\}$ and $R \in \mathcal{R}$ arbitrary. Then, for every $r, a > 0$, $K_{r+a} \leq K_r$. All these copulæ are PQD, and the limiting copula is the independent copula.

Example 11. Let $\bar{F}$ be $TTE(W,R)$, with an arbitrary $R \in \mathcal{R}$ and with W given by

$$W(x) = \left(\frac{\beta}{\beta+x}\right)^{\alpha},$$

where $\alpha > 0$. One may check that $K_{r+a} = K_r, \forall a, r \geq 0$

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