On (f,g)-out-of-$((i,j),n)$ System and its Reliability

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Abstract

The (f,g)-out-of-$((i,j),n)$:F $(f>g)$ system consists of n components ordered in a line or a circle, while the system fails if, and only if, there are at least f failed components in the system or at least g failed components among components $i, i+1, \cdots, j-1, j$ (Here the component index has module n property in the circular case, i.e., components i and $n+i$ indicate the same one, $j-i \geq g$, if $j>i$; $n+j-i+1 \geq g$, if $j<i$). In this paper, we present the system reliability formulae with the forms of recursive equations and product of matrices by means of a probability argument and a two-stage finite Markov chain imbedding approach, respectively, for the (f,g)-out-of-$((i,j),n):F$ systems.

Keywords: (f,g)-out-of-$((i,j),n):F$ system; Two-stage Markov chain imbedding approach; Reliability; Recursive equation.

1. Introduction

The existence of multiple failure criteria is a common situation for complex system, especially for consecutive-*k* type systems. The (f,g)-out-of-$((i,j),n)$:F $(f>g)$ system is one such system. It consists of n components ordered in a line or a circle, while the system fails if, and only if, there exist at least f failed components in the system or at least g failed components among components $i, i+1, \cdots, j-1, j$. Here the component index has modular n property for the circular case, i.e., components i and $n+i$ indicate the same one, $j-i \geq g$, if $j>i$; $n+j-i+1 \geq g$, if $j<i$. Throughout the paper, the independence among components is assumed. In this paper, we present the formulae for the system reliability for the (f,g)-out-of-$((i,j),n):F$ systems. It is given in the forms of recursive equation and product of matrices by means of a probability argument and a two-stage finite Markov chain imbedding approach, respectively. The latter method is emphasized in this paper. Fu (1986) first used the finite Markov chain imbedding approach for studying system reliabilities. Since then numerous authors, e.g., Fu and Hu (1987), Chao and Fu (1991), Fu and Koutras (1994), Koutras (1996), Chang, Cui & Hwang (1999,2000), Cui, Kuo & Xie (2001,2002), among others, have employed this method in the discussion of system reliabilities. The earlier studies in using this method showed that many important reliability systems can be described by some imbedded finite Markov chains. This approach can usually be employed to obtain compact reliability formulas and this approach can also be used for systems with dependent components.

2. Recursive Equation Form for System Reliability

First we discuss the linear case. Let $R_L((f,g)-(i,j);n)$ denote the reliability of (f,g)-out-of-$((i,j),n):F$ system, and $R(l,n:F)$ reliability of l-out-of-$n:F$ system. Let $p_t \equiv 1-q_t$, $(t=1,2,\cdots,n)$ be the reliability of component t.

Theorem 1. The reliability of (f,g)-out-of-$((i,j),n):F$ system can be computed as follows

$$R_L((f,g)-(i,j);n)$$
$$= \sum_{t_1=0}^{\min(i-1,f-1)} \sum_{t_2=0}^{\min(n-j,f-t_1-1)} R(t_1, i-1:F) R(t_2, n-j:F) R(\min(g, f-t_1-t_2), j-i+1:F),$$

where t_J-out-of-$l:F$ system consists of components $1,2,\cdots,i-1$ and $j+1,\cdots,n$ for $(J=1,2$ and $l=i-1,n-j)$, respectively, and $\min(g,f-t_1-t_2)$-out-of-$(j-i+1):F$ system consists of components $i,i+1,\cdots,j$.

Proof. The linear (f,g)-out-of-$((i,j),n):F$ system is equivalent to a series system consisting of three subsystems, so that the theorem is followed easily. ■

In fact, we can rearrange the components as follows without changing the system reliability: $i,\cdots,j,j+1,\cdots,n,1,\cdots,i-1$ if $i>1$ (if $i=1$, no rearrangement is needed). Thus without loss of generality, we only need to consider the linear (f,g)-out-of-$((1,k),n)$:F system.

Theorem 2. A recursive formula for the reliability of (f,g)-out-of-$((i,j),n):F$ system is

$$R_L((f,g)-(1,k);n)=p_nR_L((f,g)-(1,k);n-1)+q_nR_L((f-1,g)-(1,k);n-1),$$

with two initial conditions:

(i) $R_L((f-l,g)-(1,k);k)=R(\min(f-l,g),k:F)$;

(ii) $R_L((g,g)-(1,k);n-l)=R(g,n-l:F), l=0,1,\cdots,n-k.$

Proof. It is quit obvious, thus the proof is omitted here. ■

Similarly, by using Theorem 1, we have the following corollary.

Corollary 1. We have

$$R_L((f,g)-(1,k);n)=\sum_{t_1=0}^{\min(f-1,n-k)}R(t_1,n-k:F)R(\min(f-t_1,g),k:F),$$

where the t_1-out-of-$(n-k):F$ system consists of components $k+1,k+2,\cdots,n$.

For the circular case, we can break the circle between components $i-1$ and i to convert the circular system to a linear (f,g)-out-of-$((1,k),n):F$ system (here $k=j-i+1$ if $j>i$, otherwise $k=j+n-i+1$) with ordered components $i,i+1,\cdots,j,j+1,\cdots,n,1,2,\cdots,i-1$ in positions $1,2,\cdots,n$. We can then use the above results in linear case to obtain the system reliability for the circular case.

3. Markov Chain Representation for (f,g)-out-of-$((1,k),n):F$ System

As mentioned before, we only need to consider the linear (f,g)-out-of-$((1,k),n):F$ system instead of the general linear (f,g)-out-of-$((i,j),n):F$ system. Hence, this system is studied here.

3.1 The Linear (f,g)-out-of-$((1,k),n):F$ System

For the linear (f,g)-out-of-$((1,k),n):F$ system, we define stochastic process $X(t)=I_{\{t\le k\}}X_1(t)+I_{\{k<t\le n\}}X_2(t)$ with state space $S=S_1\bigcup S_2\bigcup\{s_N\}$, where $S_1=\{s_i^1:0\le i\le g-1\}$ (here s_i^1 indicates a working state in which the system $(1,2,\cdots,t)$, $(t\le k)$ has failed i components), and $S_2=\{s_j^2:0\le j\le f-1\}$ (here s_j^2 indicates a working state in which the system $(1,2,\cdots,t)$, $(k<t\le n)$ has j failed components), i.e., when $t\le k$, the stochastic process $X_1(t)$ takes transition on the space $S_1\bigcup\{s_N\}$; when $k<t\le n$, the stochastic process $X_2(t)$ takes transition on the space $S_2\bigcup\{s_N\}$. It is clear that $X_1(t)$ represents a g-out-of-$k:F$ system; $X_2(t)$ represents a j-out-of-n:F system. We have $N=|S|=|S_1|+|S_2|+1=g+f+1$. The initial state of the process $X_2(t)$ depends on the state of the process $X_1(t)$ at time k. The process $X(t)$ has the following transition diagram, if the virtual time k' is introduced $(k<k'<k+1)$. It is clear that the process $X(t)$ has two stages and Markov property.

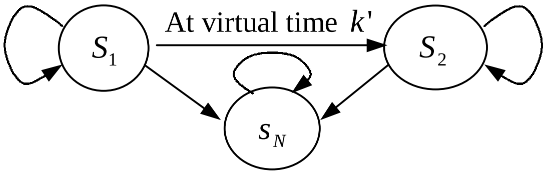

Figure 1. Transition diagram for $X(t)$

Since $X_1(t)$ is a Markov process and the transition rules are as follows.

(i) At time t, $(k < t \le n)$, $s_i^1 \to s_i^1$ with probability p_t, $s_i^1 \to s_{i+1}^1$ with probability q_t;

(ii) at time t,, $s_N \to s_N$ with probability 1.

Thus we get the probability transition matrix $\Lambda_t^1(k)$ for the process $X_1(t)$, i.e.,

$$\Lambda_t^1(k) = \begin{pmatrix} p_t & q_t & 0 & \\ 0 & \ddots & \ddots & 0 \\ & \ddots & p_t & q_t \\ & & 0 & 1 \end{pmatrix}_{(g+1)\times(g+1)} .$$

After the k th step, the process $X(t)$ goes into one state of $S_1 \cup \{s_N\}$, say s_i^1 or s_N. For the s_i^1 case, there are i failed components, and it is equivalent to the process $X_2(t)$ is at the initial state i; For the s_N case, the system fails, i.e., the process $X(t)$ stays at state s_N forever. In other words, in the virtual time k' $(k < k' < k+1)$, we have the following state transitions for the process $X(t)$.

$$s_i^1 \to s_i^2 \quad \textit{with probability } 1; \qquad s_N \to s_N \quad \textit{with probability } 1 .$$

Similarly, we have the probability transition matrices for the $X_2(t)$ as follows.

$$\Lambda_t^2(n-k) = \begin{pmatrix} p_t & q_t & 0 & \\ 0 & \ddots & \ddots & 0 \\ & \ddots & p_t & q_t \\ & & 0 & 1 \end{pmatrix}_{(f+1)\times(f+1)} .$$

By using the Chapman-Kolmogorov equation and the definition of the system reliability, we have the following theorem.

Theorem 3. The reliability for the linear (f,g) -out-of- $((1,k),n):F$ system can be computed as

$$R_L((f,g)-(1,k);n) = \left(\boldsymbol{\pi}_0^1 \prod_{t=1}^{k} \Lambda_t^1(k)\right) \mathbf{V} \left(\prod_{t=k+1}^{n} \Lambda_t^2(n-k) \mathbf{U}_0^T\right),$$

where $\boldsymbol{\pi}_0^1 = (1,0,\cdots,0)_{1\times(g+1)}$, $\mathbf{U}_0 = (1,\cdots,1,0)_{1\times(f+1)}$, $\mathbf{V} = \begin{pmatrix} \mathbf{I}_{g\times g} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 1 \end{pmatrix}_{(g+1)\times(f+1)}$, $\mathbf{I}_{g\times g}$ is a unite matrix.

Proof. Besides the statements above, the matrix $\mathbf{V}$ represents the transition for the process $X(t)$ at the virtual time k'. The proof is completed. ■

3.2. An Numerical Example

We consider a linear $(5,3)$ -out-of- $((1,10),20):F$ system with component reliabilities $p_t = 0.8 + 0.01t, (1 \le t \le 10)$ and $p_t = 0.55 + 0.02t, (11 \le t \le 20)$. We have $|S_1| = g = 3$, $|S_2| = f = 5$, $N = 9$. The transition diagram of the process $X(t)$ is as in Figure 2. By using Theorem 3, we get $R_L((5,3)-(1,10);20) = 0.77788$.

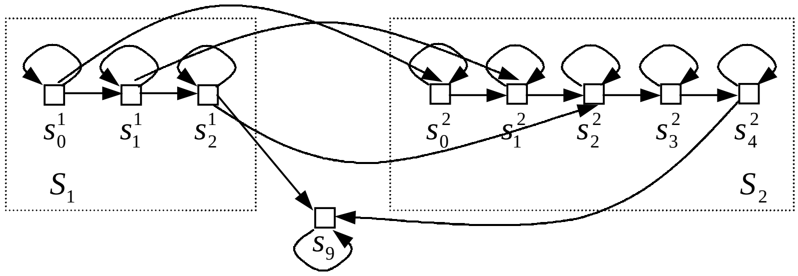

Figure 2. Transition Diagram for $X(t)$

4. Conclusion

A new reliability system, named (f, g)-out-of-$((i, j), n): F$ system, is introduced and it deals with multiple failure criteria used in real situations. The system reliability formulae are presented by recursive equations and product of matrices by using probability arguments and two-stage finite Markov chain imbedding approach. The latter method can not only be used in independent cases but also in dependent cases for many other reliability systems under a proper definition for an imbedded Markov chain.

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