Optimal Inspection and Maintenance for Stochastically Deteriorating Systems

Martin Newby
Department of Actuarial Science & Statistics
City University
LONDON EC1V 0HB
UNITED KINGDOM
m.j.newby@city.ac.uk

Richard Dagg
RABO Bank
London

Abstract

Our motivation is the problem of determining condition based maintenance policies, for systems whose degradation may be modelled by a continuous time stochastic process. The process is modelled by a Lévy process; failure occurs when the degradation reaches a critical level. The system may be inspected or repaired at any time, and that the costs of inspections and repairs may depend on the of system degradation. Optimal inspection policies for systems which may be directly and perfectly observed are determined. We establish recursive relationships and integral equations based on the Lévy and obtains optimal solutions.
Keywords: Levy process; gamma process; renewal-reward; integral equations; maintenance; replacement

1. Introduction: Maintenance and Inspection of Deteriorating Systems

We determine optimum inspection and maintenance policies for systems subject to wear and deterioration. The developments are illustrated by examples using Wiener and gamma processes Many systems can now be regarded as having high reliability and thus do not permit adequate statistical estimation of failure characteristics. In such a case other state dependent methods based on process monitoring and diagnostics are required to provide prognostics for decision-making. Examples are: crack growth (Sobcyk, 1987) as a basis for maintenance planning for airframes or offshore structures; coastal flood barriers subject to erosion (Van Noortwijk, 1996). We consider only perfect inspection. The framework developed can be extended to include different types of maintenance, imperfect inspection and covariates (Jewell and Kalbfleisch, 1996).

2. Systems subject to Stochastic Degradation

Degradation is represented by a random variable X_t, it may be a multivariate process, but assume for the moment that it is univariate. The degradation is measured on an increasing scale, $X_s < X_t$ means the system is worse at time s than at time t. A utility function $U(x)$ defines a univariate stochastic measure of degradation. Decreasing processes are simply transformed to increasing processes (Van Noortwijk 1996) by $X_t' = K - X_t$.

Most models are based on damage accumulation. Van Noortwijk (1996) points out that in systems subject to shocks, the order in which the damage (i.e. the shocks) occurs is immaterial so that the random deterioration in equal time intervals forms a set of exchangeable random variables. This also implies that the process has stationary increments. Exchangeable and stationary increments are similar to the stronger properties of stationary and independent increments of Lévy processes (Breiman, 1968). The analytical advantages of Lévy generally outweigh other shortcomings of the models, e.g. a monotonic Lévy process is a jump process (Rogers and Williams, 1994).

The start of each new cycle is a regeneration point. Retaining the Markov property lets us use the Renewal Reward Theorem (Ross, 1970) based on the expected cost and expected length of a replacement cycle. The average cost per unit time is:

$$E\left[\frac{C(t)}{t}\right] \to \frac{E[X]}{E[Y]} \quad \text{as} \quad t \to \infty$$

Where $E[X]$ and $E[Y]$ are the expected cost per cycle and the expected length of a cycle.

3. Optimal Perfect Inspection Policies

The replacement policy is assumed to be fixed in all respects, other than the inspection interval. The degradation process is $X = \{X_t \mid t \geq 0\}$ with initial level of degradation $X_0 = 0$. If this is not the case a simple transformation of the model results in a model for which the initial level of degradation is indeed zero. A number of models are defined, all the models descriptions have a similar structure and as will be seen, lead to similar mathematical formulations as integral equations.

The policy σ is 'Inspect the system every τ time units', $\sigma = \{\tau, 2\tau, 3\tau, \ldots\}$, inspections continue until the system fails or is replaced, when time returns to zero, and the inspection schedule continues. This policy is simple to understand and apply. For non-periodic inspections we consider a stationary state-dependent policy. This policy is adaptive, and the inspection intervals change as the degradation level changes. The optimal policy in this case is defined completely by the function $\tau^*(x)$ determined by dynamic programming.

MODEL: perfect inspection

1. The state-space is partitioned into intervals A_i. with $A_0 = (-\infty, s_0)$, $A_k = [s_k, s_{k+1})$, $k = 0, 1, \ldots, n-1$, $s_k < s_{k+1}$, $s_{n+1} = \infty$.
2. Inspection reveals the true state of the system.
3. If, on inspection $X_t \in A_0$, the system is not replaced and continues operating until the next inspection. Inspection incurs cost C_0.
4. If, on inspection, $X_t \in A_i$ $i \geq 2$, the system is replaced at cost C_i, and returned to its original state.
5. The system fails and is replaced at the first moment the process hits the set A_n.
6. The cost of replacing a failed system is C_n, $C_i < C_j$ for $i < j$.

In the non-monotonic process we add a supplementary process $M_t = \sup\{X_u \mid 0 \leq u \leq t\}$, the maximum value of X_t. The action after an inspection is determined by the bivariate process (X_t, M_t) (Rogers and Williams, 1994). In the monotone case we use the process X_t directly.

3.1 Expected Cost per Unit Time

Define $V(x,\tau)$ to be the (random) cost per cycle conditional an initial level of degradation level x and inspection interval τ. Similarly $L(x,\tau)$ is the (random) length of a cycle for a system under the same conditions.. V and L are random variables conditioned on x and τ, the notation is to simplify derivations. The arguments for V and L are identical and lead to equivalent integral equations.

Define the expected cost until replacement, $v(x, \tau) = \mathrm{E}[V(x, \tau) \mid X_0 = x]$, and the average time until replacement $l(x, \tau) = \mathrm{E}[L(x, \tau) \mid X_0 = x]$ with initial degradation level x, and inspection policy τ.

Expected Cost per Cycle

Using a standard dynamic programming argument, we can express the cost until replacement as the sum of the cost incurred at the next inspection and the cost incurred thereafter. X_t represents the degradation process, and M_t the maximum value of that process over the period $[0, t]$. Since we are dealing only with Lévy processes, the results are independent of the current time, and so, without loss of generality, we assume that the current time is $t = 0$, and condition all variables on the event $X_0 = x$. (1_A is the indicator function of the set A)

$$V(x,\tau) \mid X_\tau, M_\tau = \left[C_0 + V(X_\tau,\tau)\right] 1_{\{X_\tau \in A_0, M_\tau \notin A_n\}} + \sum_{i=1}^{n-1} C_i 1_{\{X_\tau \in A_i, M_\tau \notin A_n\}} + C_n 1_{\{M_\tau \in A_n\}}$$

the sum of the cost in the event of continuing (i.e. not replacing the system) and the costs for each possible replacement event. The expected cost is thus reduces to the integral equation

$$v(x) = p(x) + q(x)v(0) + \int_0^s v(y) K(y,x)\, dy$$

a Fredholm equation of the second kind.

The expected length of a cycle is determined by a similar equation

$$l(x,\tau) = \int_0^{\tau} [1 - G_\tau(h \mid x)]\, dh + l(0,\tau)\, u(x,\tau) + \int_0^{s_0} l(y,\tau) K_\tau(y \mid x)\, dy$$

Optimal Inspection Policies

We can call on the renewal reward theorem to compute the average cost. The expected average cost per unit time over an infinite time horizon, for a new system with inspection policy τ is given by

$$\mathbb{C}(0,\tau) = \frac{v(0,\tau)}{l(0,\tau)}$$

Since a new system starts with no degradation, the policy should minimises $\mathbb{C}(0,\tau)$:

$$\tau^* = \arg\inf_{\tau>0} \{\, \mathbb{C}(0,\tau) \,\}$$

3.2 Optimal Periodic Inspection: Discounted Cost Criterion

3.2.1 Discounted Total Cost

The discounted cost criterion is simpler than the previous case, and allows future costs to be discounted, a significant factor in real applications.

Define $V_\delta(x,\tau)$ to be the discounted total cost for a system with degradation x, t_i as the inter-event time (an event being either an inspection or failure), then

$$v_\delta(x,\tau) = E\big(V_\delta(x,\tau) \mid X_0 = x\big) = E\left(\sum_{n=1}^{\infty} e^{-\delta(t_1+\cdots+t_n)} C(X_{t_{n-1}}, X_{t_n}; \tau) \,\Big|\, X_0 = x \right)$$

is the expected discounted total cost, where $C(x,y,\tau)$ represents the (random) cost incurred if the system is in state y at time t_n when it was in state x at time t_{n-1}, with inspection policy is τ.

Again

$$v_\delta(x,\tau) = c_\delta(x,\tau) + v_\delta(0,\tau) u_\delta(x,\tau) + \int_0^{s_0} v_\delta(y,\tau) K_\tau(y \mid x)\, dy \qquad (1)$$

The optimal policy in this case is given by

$$\tau_\delta^* = \arg\inf_{\tau>0} \{\, v(0,\tau) \,\}$$

Non-periodic inspection is more general and often more useful than the previous case. We define the function v_δ to be the value function for the δ-optimal policy, i.e.

$$v_\delta(x) = \inf_\pi v_\pi(x)$$

For the general cost function given by equation (1) above.

3.3 Numerical Results and Comments on the model

Let us now consider the numerical results found earlier. In the three examples the underlying assumptions are:

- The degradation process is a gamma process with $\alpha/\beta = 1$. We vary the parameter α to look at the effects of changing the variability of the underlying process, for a given average rate of degradation
- The failure limit of the process $c = 1$. We vary the replacement limit r to see the effect of changing this limit.
- The cost of an inspection C_I is 1 unit. We vary the replacement cost C_R and failure cost C_F, to look at changes in the optimal policies.

The optimal periodic policy in these cases is determined by the inspection period.

Properties of the cost function for Periodic Policies

We now consider the form of the cost function under periodic inspection. For the model we are using we have found three distinct shapes for the cost function: Single Local Minimum (Fig 1a); Double Local Minimum (Fig 1b); Monotone Decreasing (Fig 1c).

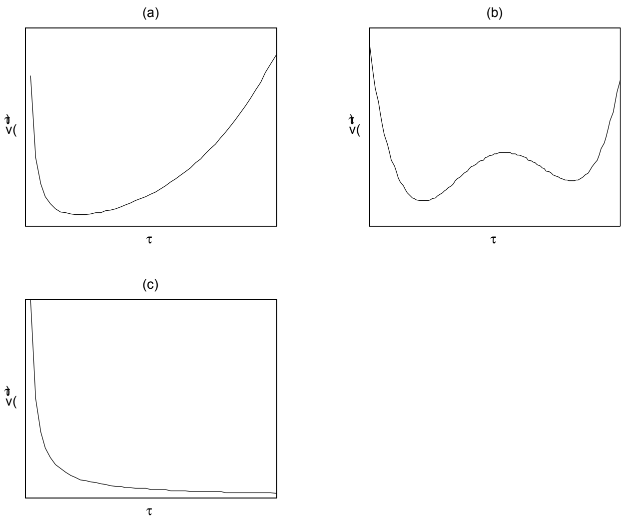

Figure 2: cost functions

4. Conclusion

We have presented a systematic approach to the determination of optimal inspection intervals for systems whose degradation follows a Lévy process, and fail when this degradation reaches a given threshold level. The approach rests on the application of the renewal reward theorem and on dynamic programming techniques.

The main example considered is that of a gamma process. The results of the model provide sensible and realistic inspection policies for such systems, and gives insight into the behaviour of the system and the effect of applying various inspection policies.

The model applies only to a simple Lévy process model of degradation, but can readily be extended to a generalised gamma process as defined by Van-Noortwijk (1996).

This paper outlines an approach and structure which will be extended later and applied to include covariates, imperfect inspections, and different repair policies. The model here shows the feasibility of this programme.

5. References

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