Some Comparisons Between Generalized Order Statistics

M. Carmen Ruiz
Dept Mat. Apl. y Estadística
Universidad Politécnica de Cartagena
30203, Murcia
Spain
maricarmen.ruiz@upct.es

Manuel Franco, José M. Ruiz
Dept Estadística e I.O.
Universidad de Murcia
30100, Murcia
Spain
mfranco@um.es; jmruizgo@um.es

Abstract

Ordinary order statistics and record values, among others, are included in the model of generalized order statistics introduced in Kamps (1995). In this paper, we establish stochastic comparisons among spacings of generalized order statistics, which extend and complete some known results in the literature.

1 Introduction

Order statistics and record values are widely used in statistical modeling and inference, because both models describe random variables (r.v.) arranged in order of magnitude. In Reliability, an order statistic represents the life-length of some r-out-of-n system whereas record values are closely connected with the occurrence times of some corresponding non-homogeneous Poisson process and used in shock models.

In addition to these well known models, several other models of ordered r.v. are included in the model of generalized order statistics (gOS's) introduced by Kamps (1995), such as order statistics with non-integral sample size, sequential order statistics, k-th record values, Pfeifer's record model, k_n-record from non-identical distributions and ordered r.v. which arise from truncation of distributions.

These models can be applied in reliability theory. For instance, sequential order statistics are an extension of ordinary order statistics and serve as a model describing certain dependencies or interactions among the system components caused by failures of components. Likewise, the di¤erent types of record values mentioned above extend the model of ordinary record values and they are more applicable to practical situations.

Several papers are devoted to the study of orderings among order statistics or record values, in order to compare the reliability of di¤erent systems and certain aging processes; see for instance Gupta and Kirmani (1988), Kochar (1990) and (1996), Shaked and Shanthikumar (1994), Raqab and Amin (1996) and Belzunce, Franco, Ruiz and Ruiz (2001).

Particularly, some authors have obtained comparison results among spacings of order statistics and record values. In this setting, Kochar and Kirmani (1995) compare, in the stochastic and failure rate orders, normalized spacings of order statistics from a random sample with DFR distribution. In the case of two samples, Kochar (1999) and Khaledi and Kochar (1999) establish comparisons among spacings of order statistics when the populations are ordered and one of them belongs to a certain aging class. Besides, Kochar (1990) gives su¢cient conditions to compare spacings of record values in the stochastic order.

The purpose of this paper is to establish some comparison results between normalized spacings of gOS's formed from two distributions F and G, which allow us to extend the above results to several other models (sequential order statistics, record values, k-th record values, ...). Furthermore, we can compare spacings of di¤erent models, i.e., between ordinary order statistics and sequential order statistics, between record values and Pfeifer's record values, etc.

2 Definitions

First, we give the definition of the r-th gOS based on a distribution function F using its marginal density function, which was introduced by Kamps (1995).

Let the function $g_m(x)$, $m \in \mathbb{R}$ be defined as follows

$$g_m(x) = \begin{cases} \frac{1}{m+1}\left(1-(1-x)^{m+1}\right) & \text{if } m \neq -1 \\ \log\left(\frac{1}{1-x}\right) & \text{if } m = -1 \end{cases} \quad \text{for all } x \in [0,1) \tag{1}$$

Definition 1 Let $n \in \mathbb{N}$, $1 \le r \le n$, $k \ge 1$, $m_1, \ldots, m_{n-1} \in \mathbb{R}$, $M_s = \sum_{j=s}^{n-1} m_j$, $1 \le s \le n-1$, be parameters such that $\gamma_s = k + n - s + M_s \ge 1$ for all $s \in \{1, \ldots, n-1\}$, $m_1 = m_2 = \ldots m_{r-1} = m$, and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \ge 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary if $n = 1$). The marginal density function of the r-th gOS $X_{(r,n,\tilde{m},k)}$ based on F is given by:

$$f_{X_{(r,n,\tilde{m},k)}}(x) = \varphi_{r:n}(F(x))f(x) \tag{2}$$

where $\varphi_{r:n}(x) = \frac{c_{r-1}}{(r-1)!}(1-x)^{\gamma_r-1}g_m^{r-1}(x)$, $x \in (0,1)$ and $c_{r-1} = \prod_{i=1}^{r}\gamma_i$.

As we have already commented, several interesting models are included in the model of gOS's (see Kamps (1995). For example, if $m_1 = \ldots = m_{n-1} = 0$ and $k = 1$ (i.e. $\gamma_r = n - r + 1$, $1 \le r \le n-1$), Eq. (2) reduces to the density of the r-th ordinary order statistic of a random sample with distribution function F. And choosing $m_1 = \ldots = m_{n-1} = -1$ and $k \in \mathbb{N}$ (i.e. $\gamma_r = k$, $1 \le r \le n-1$), we obtain the model of the k-th record values based on the sequence $(X_i)_{i \in \mathbb{N}}$ of independent and identically distributed (i.i.d.) r.v. with distribution function F.

3 Orderings between gOS's

In this section, we first study conditions on the parameters to compare gOS's based on the same distribution, and then based on two different distributions. As consequences, we obtain the preservation of the stochastic, failure rate and likelihood ratio ordering under the formation of gOS's, which extend the preservation results of other authors.

For that, we will denote $\tilde{m} = (m_1, \ldots, m_{n_1-1})$, $m_1 = \ldots = m_{r_1-1} = m$, $\tilde{m}' = (m'_1, \ldots, m'_{n_2-1})$, $m'_1 = \ldots = m'_{r_2-1} = m'$, $\gamma_{r_1} = k_1 + n_1 - r_1 + M_{r_1}$ and $\gamma'_{r_2} = k_2 + n_2 - r_2 + M'_{r_2}$.

Theorem 1 Let $X_{(r_1,n_1,\tilde{m},k_1)}$ and $X_{(r_2,n_2,\tilde{m}',k_2)}$ be gOS's based on an absolutely continuous distribution function F. Then, $X_{(r_1,n_1,\tilde{m},k_1)} \le_{LR} X_{(r_2,n_2,\tilde{m}',k_2)}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge -1$.

Remark 1 Kamps (1995) proved the last theorem when $\tilde{m} = \tilde{m}'$.

In particular, Theorem 1 extends some results of Bapat and Kochar (1994) and Raqab and Amin (1996) on ordinary order statistics, Gupta and Kirmani (1988) and Kochar (1990) and (1996) on record values, and Raqab and Amin (1996) on k-record values.

Now, we give three theorems on preservation of orderings under the formation of gOS's with different parameters and based on F and G, which extend the results of Singh and Vijayasree (1991) and Raqab and Amin (1996) on ordinary order statistics.

Theorem 2 If $F \le_{ST} G$, then $X_{(r_1,n_1,\tilde{m},k_1)} \le_{ST} Y_{(r_2,n_2,\tilde{m}',k_2)}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge -1$.

Theorem 3 If $F \le_{HR} G$, then $X_{(r_1,n_1,\tilde{m},k_1)} \le_{HR} Y_{(r_2,n_2,\tilde{m}',k_2)}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge -1$.

Theorem 4 If $F \le_{LR} G$, then $X_{(r_1,n_1,\tilde{m},k_1)} \le_{LR} Y_{(r_2,n_2,\tilde{m}',k_2)}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge 0$.

4 Orderings between spacings of gOS's

In this section, we establish stochastic comparisons of spacings of gOS's from restricted families of distributions.

For that, we define normalized spacings of gOS's from two absolutely continuous distribution functions F and G by, $U_{r_1:n_1} = \gamma_{r_1}\left(X_{(r_1,n_1,\tilde{m},k_1)} - X_{(r_1-1,n_1,\tilde{m},k_1)}\right)$ and $V_{r_2:n_2} = \gamma'_{r_2}\left(Y_{(r_2,n_2,\tilde{m}',k_2)} - Y_{(r_2-1,n_2,\tilde{m}',k_2)}\right)$ for $2 \le r_1 \le n_1$ and $2 \le r_2 \le n_2$.

Now, we give sufficient conditions to compare conditioned spacings of gOS's in the failure rate order.

Theorem 5 Let $F \le_{HR} G$ and either F or G be DFR. Then, for $\gamma_{r_1} \ge \gamma'_{r_2}$ and $y \ge 0$

$$U_{y;r_1:n_1} = \left(U_{r_1:n_1} \mid X_{(r_1-1,n_1,\tilde{m},k_1)} = y\right) \le_{HR} \left(V_{r_2:n_2} \mid Y_{(r_2-1,n_2,\tilde{m}',k_2)} = y\right) = V_{y;r_2:n_2}$$

The following theorem establish comparisons in the stochastic order between spacings of gOS's, when the distribution functions are ordered in the hazard rate order and one of them is supposed to have a decreasing failure rate function.

Theorem 6 Let $F \le_{HR} G$ and either F or G be DFR. Then $U_{r_1:n_1} \le_{ST} V_{r_2:n_2}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge -1$.

Note that if $\tilde{m} = (0, \ldots, 0)$, $\tilde{m}' = (0, \ldots, 0)$, $k_1 = k_2 = 1$, we obtain Theorem 1.2 of Kochar (1999) and Theorem 2.1 of Khaledi and Kochar (1999) on ordering of spacings of ordinary order statistics, and if $\tilde{m} = (-1, \ldots, -1)$, $\tilde{m}' = (-1, \ldots, -1)$, $k_1 = k_2 = 1$, we obtain Corollary 3.2 of Kochar (1990) on record values.

Analogously, in the next theorem we establish comparisons in the stochastic order between spacings of gOS's, when the distribution functions are ordered in the reversed hazard rate order and one of them has increasing reversed failure rate function.

Theorem 7 Let $F \le_{RH} G$ and either F or G be IRF. Then $W_{r_1:n_1} \le_{ST} Z_{r_2:n_2}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge -1$, where $P(W \le t) = \overline{G}(-t)$ and $P(Z \le t) = \overline{F}(-t)$.

In particular, we obtain a result for spacings of ordinary order statistics, which is a particular case of Corollary 3.1 of Hu and Wei (2001) when $j = i + 1$, $q = p + 1$, $p \ge i$ and $p - i \ge m - n$.

Let us see now comparisons between spacings of gOS's when their population are ordered in the likelihood ratio ordering.

Theorem 8 Let $F \le_{LR} G$ and either F or G be DFR. Then $U_{r_1:n_1} \le_{HR} V_{r_2:n_2}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge 0$.

Note that Theorem 2.1 of Kochar (1999) and Theorem 2.2 of Khaledi and Kochar (1999) can be obtained from Theorem 8 taking $\tilde{m} = (0, \ldots, 0)$, $\tilde{m}' = (0, \ldots, 0)$, $k_1 = k_2 = 1$.

Analogously, we get the next result.

Theorem 9 Let $F \le_{LR} G$ and either F or G be IRF. Then $W_{r_1:n_1} \le_{HR} Z_{r_2:n_2}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge 0$, where $P(W \le t) = \overline{G}(-t)$ and $P(Z \le t) = \overline{F}(-t)$.

In particular, we obtain a corollary for spacings of ordinary order statistics and the following result.

Corollary 10 Let $F \le_{LR} G$ and either F or G be DFR. Then $U_{r_1:n_1} \le_{DISP} V_{r_2:n_2}$ for all $r_2 \ge r_1$, $\gamma_{r_1} \ge \gamma'_{r_2}$ and $m \ge m' \ge 0$.

Hu and Wei (2001) establish some comparison results between normalized spacings of ordinary order statistics, which we extend to the model of gOS's in the following theorem.

Theorem 11 If there exists an exponential random variable Z with distribution function H_Z such that $F \leq_{HR} H_Z \leq_{HR} G$, then $U_{r_1:n_1} \leq_{HR} V_{r_2:n_2}$ for all parameters $(r_1, n_1, \tilde{m}, k_1)$ and $(r_2, n_2, \tilde{m}', k_2)$ such that $(m+1)^{r_1-1} \cdot (m'+1)^{r_2-1} \geq 0$.

Finally, we give sufficient conditions to preserve the likelihood ratio ordering under the formation of spacings of gOS's.

Theorem 12 Let $F \leq_{LR} G$ and either F or G be DLR. Then, $U_{r_1:n_1} \leq_{LR} V_{r_2:n_2}$ for all $r_2 \geq r_1$, $\gamma_{r_1} = \gamma'_{r_2}$ and $m \geq m' \geq 0$.

Once again, Theorem 2.3 of Khaledi and Kochar (1999) can be obtained from our Theorem 12 taking $\tilde{m} = (0, \ldots, 0)$, $\tilde{m}' = (0, \ldots, 0)$, $k_1 = k_2 = 1$.

Remark 2 The results given in this section also hold for simple spacings of gOS's $\tilde{U}_{r_1:n_1} = X_{(r_1,n_1,\tilde{m},k_1)} - X_{(r_1-1,n_1,\tilde{m},k_1)}$ and $\tilde{V}_{r_2:n_2} = Y_{(r_2,n_2,\tilde{m}',k_2)} - Y_{(r_2-1,n_2,\tilde{m}',k_2)}$.

Acknowledgments

This work was partially supported by the Ministerio de Ciencia y Tecnología, Dirección General de Investigación under Grant BFM2000–0362.

References

Bapat, R.B. & Kochar, S.C. (1994). On likelihood rate ordering of order statistics. Linear Algebra Applied 199, 281-291.

Belzunce, F., Franco, M., Ruiz, J.M. & Ruiz, M.C. (2001). On partial orderings between coherent systems with different structure. Probab. Engineering and Informational Sciences 15, 273-293.

Gupta R.C. & Kirmani, S.N.U.A. (1988). Closure and monotonicity properties of nonhomogeneous Poisson processes and record values. Probab. Engineering and Informational Sciences 2, 475-484.

Hu, T. & Wei, Y. (2001). Stochastic comparisons of spacings from restricted families of distributions.Artificial Intelligence 29 (3), 241–288.

Pearl, J. (1986). Fusion propagation and structuring in belief networks. Stat. Probab. Letters 53, 91-99.

Kamps, U. (1995). A concept of generalized order statistics. Stuttgart: Teubner.

Khaledi, B.E. & Kochar, S. (1999). Stochastic orderings between distributions and their sample spacings-II. Stat. Probab. Letters 44, 161-166.

Kochar, S.C. (1990). Some partial ordering results on record values. Comm. Statist. Theory Meth. 19, 299-306.

Kochar, S.C. (1996). A note on dispersive ordering of record values. Calcutta Statist. Assoc. Bull. 46, 63-67.

Kochar, S.C. (1999). On stochastic orderings between distributions and their sample spacings.Stat. Probab. Letters 42, 345-352.

Kochar, S.C. & Kirmani, S.N.U.A. (1995). Some results on normalized spacings from restricted families of distributions. J. Statist. Plan. Inf. 46, 47-57.

Raqab, M.Z. & Amin, W.A. (1996). Some ordering results on order statistics and record values. IAPQR Transactions 21, 1-8.

Shaked, M. & Shanthikumar, J.G. (1994). Stochastic orders and their applications. New York: Academic Press.

Singh, H. & Vijayasree, G. (1991). Preservation of partial orderings under the formation of k-out-of-n systems of i.i.d. components. IEEE Transaction on Reliability 40, 273-276.